Implicit Equilibrium Dynamics*

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Abstract

We discuss the problem known in economics as “backward dynamics (BD)” occurring in models of perfect foresight, intertemporal equilibrium described mathematically by implicit difference equations. In a previously published paper (Medio and Raines, 2007) we showed that by means of certain mathematical methods and results known as “inverse limits theory (ILT)” it is possible to establish a correspondence between backward dynamics of a non-invertible map and forward dynamics of a related, invertible map acting on an appropriately defined space of sequences, each of whose elements corresponds to an intertemporal equilibrium. We also proved the existence of different types of topological attractors for one-dimensional models of overlapping generations. In this paper, we provide an extension of those results, constructing a Lebesgue–like probability measure on spaces of infinite sequences that allows us to distinguish typical from exceptional dynamical behaviors in a measure–theoretical sense, thus proving that all the topological attractors considered in MR07 are also metric attractors. We incidentally also prove that the existence of chaos (in the Devaney–Touhey sense) backward in time implies (and is implied by) chaos forward in time.

Keywords: Backward dynamics, Overlapping Generations, Inverse Limits, Attractors, Lebesgue-like measure

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1. **Introduction and motivation.** Dynamical system theory (DST) has been extensively and successfully applied to the analysis of economic problems. In this area of research, however, the use of mathematical notions and results is not always straightforward and many delicate questions of economic interpretation arise. To illustrate this point, let us consider two basic types of (discrete–time) models in economic dynamics: optimal growth and intertemporal economic equilibrium (IEE) models. Under generally assumed conditions, optimal growth models can, at least in principle, be represented mathematically by systems of difference equations with the canonical form:

$$x_{t+1} = F(x_t)$$ (1)

where the map $F$ is derived from the solution of a constrained intertemporal optimization problem. Iterations of $F$ generate sequences of optimal values of the state variable $x$, e.g. the capital stock, *forward in time* from any arbitrarily given initial conditions to an arbitrarily distant future. The methods of DST can therefore be employed to discuss the properties of the orbits of system (1), such as existence and stability of stationary states, existence of other interesting invariant sets and their interpretation, dependence of dynamical solutions on the system parameters, etc.

The situation is more complicated for IEE models characterized by infinite horizon and an infinity of agents and commodities, the most popular family of which is given by the overlapping generations (OLG) models. Assuming perfect foresight, here the typical mathematical formulation is a system of *implicit* difference equations like:

$$H(x_t, x_{t+1}) = 0$$ (2)

where $x$ is again a vector of state variables and the function $H$ depends on the economic fundamentals, typically utility or production functions. It is sometimes (but not generally) possible to invert $H$ with respect to $x_{t+1}$ *globally*, and therefore translate (2) into an explicit difference equation like (1). Unfortunately, in economic models this procedure may not be possible, nor are there any *a priori* justifications for assuming that it is.
When a map like $F$ cannot be properly defined, it may still be possible to invert $H$ with respect to $x_t$ globally and write a “backward dynamical system” like

$$x_t = G(x_{t+1})$$

(3)

In this case, for each given state $x_t$ at some time $t$, instead of a function $F$ determining the next state $x_{t+1}$, we have a non-invertible function $G$ that determines the previous state $x_{t-1}$. Thus, the model does not actually predict the future state from the present, but gives a (possibly empty) set $G^{-1}(x_t)$ of admissible future states. In what follows, we will label maps like $G$ “BD maps”.

The possibility that the mathematical representation of IEEs may yield “backward dynamical systems” is mentioned in early discussions of OLG models (see, for example, Gale (1973), Benhabib and Day (1982)) but, on the whole, it has surprisingly been given little attention in economic literature. In the presence of BD maps, the approach most commonly adopted is first to locate a steady state solution (a fixed point of the relevant map), next to invoke the implicit function theorem and invert the map around the fixed point and finally to perform some local analysis of the system (see, for example, Gale (op.cit., pp. 24–25), Grandmont (1989 pp. 51–52)). Although this strategy has produced a number of useful results, it restricts the investigation severely, implicitly leaving out many, possibly most orbits moving forward in time and compatible with the assumed dynamical rules, i.e., disregarding many interesting cases of IEE.

An obvious feature of systems characterized by BD maps like (3) is that there is no one-to-one correspondence between initial conditions (points in the state space) and orbits forward-in-time. On the contrary, there may be infinitely many such orbits starting from the same point of the state space and converging forward-in-time to many different sets. Consequently, in order to provide a complete characterization of those orbits and the corresponding IEEs, we need to construct a “larger” space, each of whose elements corresponds to a unique forward orbit. We must also define a map acting on that space such that its iterations can be interpreted as the dynamics forward in time for the original problem. There exists an area of research in mathematics called
inverse limits theory (ILT) that provides a natural framework for this problem. In the present context, the basic idea behind ILT is straightforward: we move from the original state space, typically a subset of $\mathbb{R}^n$, to a more complex space generated by the solutions of the implicit difference equations of the model. In so doing, we transform a non-invertible BD map, such as $G$ of equation (3), into an invertible map which is sometimes called the “natural extension” of the original map $G$.

Inverse limit theory is a body of mathematical notions and methods developed in the last seventy years or so, but relatively unknown in economics. Medio (1998) provides a preliminary analysis of some of the problems discussed in this paper, in terms of a “natural extension” of non-invertible maps. A more thorough and rigorous investigation of backward dynamical systems in economics is provided by Medio and Raines (2006, 2007). (The latter paper, of which the present one is an extension, will be labeled henceforth “MR07”). A few other notable recent works applying inverse limit spaces to the problem of backward dynamics in economics include Kennedy, Stockman and Yorke (2007, 2008) and Kennedy and Stockman (2008). A different approach to the problem of backward dynamics can be found in Gardini, Hommes, Tramontana and de Vilder (GHTV, 2009), where forward equilibria are defined as sequences where at each step equilibrium selection is determined by a random sunspot sequence. They then apply the method of Iterated Function Scheme (IFS) to a one- and a two-dimensional version of the overlapping generations model and show that, if the backward dynamics of such a model are chaotic and have a homoclinic orbit there exists an appropriately restricted IFS whose orbits converge to a fractal attractor. GHTV’s approach to the problem of backward dynamics is different from the one adopted in the present paper, both economically and mathematically. Specifically, in MR07 and here we are concerned with perfect foresight, deterministic equilibrium dynamics whereas GHTV’s main result concerns stochastic (sunspot) equilibria.

In order not to overburden our presentation and to avoid useless repetitions, in what follows we will limit our discussion to those aspects of ILT strictly necessary for the present application. For a more general discussion of ILT and its relevance to economic theory, we refer the reader to our MR07 article (specifically to its technical
Appendix A) and the extended bibliography therein.

The basic ideas and methods presented and applied in this paper, as well as some general results proved in Section 3 are relevant to a wide class of implicitly defined, discrete-time dynamical systems arising in economics. However, the main focus of the paper is on one-dimensional, overlapping generations (OLG) models. For those models, we define a small number of basic cases mathematically and economically and, for each of them, we distinguish between “typical or generic” and “exceptional” dynamical behavior. The rest of the paper is organized as follows. In Section 2 we describe a basic OLG model used as a benchmark in the rest of the paper. Section 3 provides an introductory discussion of the basic notions and methods of ILT used in the sequel. In Section 4, we argue that the (metric) attractors of a certain homeomorphism derived from the original, backward-moving map of the model can be used to identify the “typical or generic” forward-in-time orbits implicitly defined by the model. Section 5 deals with the class of OLG models represented by unimodal maps and defines three basic subclasses. In Section 6 we discuss the existence of metric attractors for a special OLG model. In Section 7, we construct a Lebesgue-like probability measure on the space of intertemporal equilibria and, in Section 8 we use this measure to prove the existence of metric attractors for the main cases of OLG models with backward dynamics. Section 9 sums up the paper.

2. The basic model

We begin our discussion showing how backward dynamics may arise in a basic, one-dimensional OLG.

For this purpose, we have chosen a slightly modified version of the “leisure-consumption” model used by Grandmont in his much-quoted (1985) analysis of endogenous business cycles (see also Grandmont (1983) and (1989)). Since the model is exceedingly well-known and its use in this paper is only instrumental, we limit its presentation to what is necessary to the understanding of our main argument, omitting many technical details.

Basic hypotheses and notation of the model are as follows:
H1, Demography: a constant population of individuals (identical except for their age), living two periods of time, and divided at each period into two equally numerous classes, respectively labeled “young” and “old”.

H2, Consumption: at each period $t$, the young agent consumes a quantity $c_t \geq 0$ of the unique perishable good, and a quantity of leisure $(\bar{l} - l_t) \geq 0$, where $\bar{l} \geq 0$ denotes the constant labor endowment and $l_t \geq 0$ is labor supply. The corresponding quantities for the old agent are $\kappa_t \geq 0, \bar{\iota} \geq 0$ and $\iota_t \geq 0$.

H3, Production: production takes place by means of current labor only; output is traded at a price $p_t$, and the wage rate is $w_t$; physical units of measure of output and labor are normalized so that one unit of labor yields one unit of output. In this case, profit maximization implies that $w_t = p_t$, $\forall t$.

H4, Preferences: for each generation living through the periods $(t, t+1)$ preferences are defined by the following utility functions:

$$v[c_t, (\bar{l} - l_t)]; u[\kappa_{t+1}, (\bar{\iota} - \iota_{t+1})]$$ (4)

where $v$ and $u$ are smooth, strictly increasing in each argument and concave.

H5, Perfect foresight: for each $t$, the young agent’s expectations concerning the values of the variables at $t + 1$ are perfectly fulfilled.

H6, Maximization: At each $t$, the young agent chooses the present and future levels of consumption and labor supply as functions of the observed current price ($= $ wage rate) and the perfectly anticipated future price $p_{t+1}$ ($= $ perfectly anticipated wage rate), subject to a two–period budget constraint.

H7, Market clearing condition: at each $t$, supply $(l_t + \iota_t)$ of, and demand $(c_t + \kappa_t)$ for the consumption good are equal.

The problem can be described in terms of the young agent’s first–period excess demand ($= $ dissaving), $z_t = c_t - l_t$, and the same agent’s second–period excess demand, $\zeta_{t+1} = \kappa_{t+1} - \iota_{t+1}$. Thus, the market clearing condition reduces to $z_t = -\zeta_t \forall t$. The maximizing problem can be represented formally, as follows:
\[
\max \{V(z_t) + U(\zeta_{t+1})\}
\]
\[
s.t. \; p_t z_t + p_{t+1} \zeta_{t+1} \geq 0 \quad \text{budget constraint (P1)}
\]
\[
z_t \geq -\bar{l}, \; \zeta_t \geq -\bar{\zeta} \; (\text{equivalently, } z_t \leq \bar{l}; \; \zeta_t \leq \bar{\zeta})
\]

where the functions \( V \) and \( U \) are derived, respectively, from the basic utility functions \( v \) and \( u \) of eq. (4), from which they inherit the fundamental properties\(^3\).

From the first order conditions of (P1), we deduce that, for each given pair of labor endowments \((\bar{l}, \bar{\zeta})\), the optimal, current and future, consumption and labor supply must satisfy the equation:

\[
H(z_t, \zeta_{t+1}) = V(z_t) + U(\zeta_{t+1}) = 0 \tag{5}
\]

where \( V(z_t) = V'(z_t)z_t \) and \( U(\zeta_{t+1}) = U'(\zeta_{t+1})\zeta_{t+1} \).

Using the market clearing requirement, we can transform (5) into an implicit difference equation like (2) in a single variable \((z\) or, equivalently, \(\zeta)\). Whether we can also obtain an explicit discrete–time dynamical system, moving forward or backward in time, depends on the properties of the functions \( V \) and \( U \) and the endowments.

There are two basic cases, depending on the (derived) utility functions \( U, V \) and the labor endowments\(^4\).

(i) Classical case: the young agent is “impatient” and wants to consume more and/or work less, borrowing from the old agent in the first period and paying back to next generation’s young agent in the second period.

(ii) Samuelson case: the young agent is thrifty and saves in the first period, lending to the old agent, so as to be able to consume more and/or work less in old age.

From the assumed properties of the utility functions, it follows that, in the classical case, the function \( U \) is monotone and therefore we can invert the implicit function \( H \) with respect to \( \zeta_{t+1} \) and, using the market clearing condition, obtain the equation

\[
z_{t+1} = F(z_t) \tag{6}
\]
where \( z_t \in [0, \bar{z}) \) and \( F(z_t) = -U^{-1}[-V(z_t)] \), whose iterations move forward–in–time. In the classical case, the function \( F \) may or may not be invertible. If it is not, the dynamics of (6) may be very complicated as shown in Benhabib and Day’s (1982) pioneering investigation of endogenous cycles and chaos in OLG models.

Vice versa, in the Samuelson case, for which \( z_t = -\zeta_t < 0 \forall t \), the implicit function \( H \) can be inverted globally with respect to \( z_{t+1} \), obtaining the equation

\[
\zeta_t = G(\zeta_{t+1})
\]

where \( \zeta_t \in [0, \bar{\zeta}) \) and \( G(\zeta_{t+1}) = -V^{-1}[-U(\zeta_{t+1})] \), whose iterations move backward in time. With a slight abuse of wording, henceforth we refer to the maps \( F \) and \( G \) as “offer curves”.

If the agent’s second period utility function \( U \) is such that the risk aversion \( R_U(\zeta) = -U''(\zeta)\zeta/U'(\zeta) < 1 \) (substitution effect prevails) for small values of \( \zeta \) and the opposite is true for larger value, the offer curve \( G \) is non–invertible. This is the case discussed by Grandmont (1985) and it gives rise to the problem of backward dynamics on which we focus in this paper.

3. Inverse limit space and admissible orbits

As we shall see in what follows, BD maps like (3) and (7) are commonly characterized by the fact that their implicitly defined forward–in–time dynamics includes many, even uncountably many different types of dynamical behavior (e.g., periodic dynamics of many different periods or chaotic dynamics). In this case, we would like to have rigorous criteria for identifying the behaviors which are “typical, or generic” and therefore likely to be observed, and those which are “exceptional” and therefore negligible.

In this Section, we argue that this problem can be thoroughly investigated by characterizing the set of forward admissible orbit (i.e., the set of all IEEs) as an inverse limit space and applying to it certain powerful results of inverse limit theory.

Consider the Samuelson OLG model described by equation (7) and assume that \( G \) is not globally invertible. We will start with the following definition:
Definition 1. An infinite sequence \( \{\zeta_t\} \) generated by a BD difference equation like (7) is said to be \textbf{forward admissible} if for each pair \((t, t + 1)\) \(\zeta_{t+1} \in G^{-1}(\zeta_t)\), and for all \(t \in \mathbb{N}, 0 \leq \zeta_t \leq \bar{\zeta}\).

The admissibility of sequences \( \{z_t\} \) can be defined similarly. In economic terms, Definition 1 restricts the sequences of excess demand to those that satisfy the requirement of intertemporal maximization under constraint and market equilibrium, i.e., infinite, forward admissible sequences correspond to \textbf{intertemporal, perfect foresight competitive equilibria (IEE)}. Notice that the logic of the model requires that admissible sequences be \textit{infinite}. To see this suppose that, at time \(T\), no value \(\zeta_{T+1}\) exists such that \(\zeta_{T+1} = G^{-1}(\zeta_T)\), i.e., the set \(G^{-1}(\zeta_T)\) is empty. Economically, that means that \(\zeta_T\) will not be realized since the young agent at time \(T\) will not decide to save an amount \(-z_T\) because this would only be justified by the (perfectly foreseen) expectation of a positive excess demand in his/her old age equal to \(\zeta_{T+1} \in G^{-1}(\zeta_T)\). But if \(\zeta_T = -z_T\) are not realized, by the same token nor will be \(\zeta_{T-1} = -z_{T-1}\) and so on and so on, back in time all the way to the initial value. In short, no finite sequence can satisfy the requirements of IEE.

![The Samuelson case](image)

Figure 1. A non–invertible offer curve

Figure 1, depicting a Samuelson OLG model with a non–invertible, unimodal
offer curve $G$, shows an example of an interrupted, non-admissible forward-in-time sequence.

As the diagram shows, the set $G^{-1}(\zeta)$ is empty for all $\zeta > \zeta_{\text{MAX}} = G(\zeta^*)$. Thus, for example no sequence including the sub-sequence $\zeta_1, \zeta_2, \zeta_3$ can be continued and therefore it cannot be admissible.

There is a simple way to include only admissible sequences in our construction. Let $X \subset \mathbb{R}^+$ be the domain of $G$, and let $I = \bigcap_{n \geq 0} G^n(X)$. In our case, it is easy to verify that, if $\zeta^*$ is the “critical value” of $\zeta$ for which $G'(\zeta) = 0$ and $\zeta_{\text{MAX}} = G(\zeta^*)$, then $I = [0, \zeta_{\text{MAX}}]$ is $G$-invariant and the restriction $G|_I$ is a surjection, so that, for $\zeta \in I, G^{-1}(\zeta)$ is never empty. Clearly, if $G$ is surjective on $X$ to begin with, then $I = X$.

The next step is to provide a proper characterization of the space of all forward admissible sequences. For this purpose, we will make use of a body of mathematical concepts and method known as ”inverse limit theory” (ILT). Since we discussed it in great detail in our above-quoted article MR07, and in particular its Appendix A, we will describe here only the essential parts and for a more exhaustive discussion we refer the reader to that article and the references therein. Consider a sequence $X_1, X_2, \ldots$ of metric spaces (called factor spaces) and a sequence $f_1, f_2, \ldots$ of continuous functions (called bonding maps) such that, for each $i \in \mathbb{N}, f_i : X_{i+1} \rightarrow X_i$. The double sequence $\{X_i, f_i\}$ is called an inverse sequence\(^5\). The subset of the product space $\prod_{i=1}^{\infty} X_i$ to which the point $(x_1, x_2, \ldots)$ belongs if and only if $f_i(x_{i+1}) = x_i \forall i \in \mathbb{N}$ is called the inverse limit of the inverse sequence $\{X_i, f_i\}$, and is denoted by $\lim_{\leftarrow} \{X_i, f_i\}$. If the factor spaces are metric spaces, so is the inverse limit space derived from them. More specifically, if $d_i$ is a metric on $X_i$ bounded by 1, we can define an induced metric $\hat{d}$ on $\lim_{\leftarrow} \{X_i, f_i\}$ as follows:

$$\hat{d}(\hat{x}, \hat{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^{i-1}} \quad (11)$$

where $\hat{x} = (x_1, x_2, x_3, \ldots)$ and $\hat{y} = (y_1, y_2, y_3, \ldots)$. Thus, on that space certain topological notions (such as closed, open, dense set, etc.), which we need in our
investigation, are well defined.

In the case in which there is a single factor space $X$ and a single bonding map $f: X \to X$, the inverse limit space is simply denoted as $\varprojlim \{X, f\}$. We are mostly concerned with this case, to which we refer as “simple” inverse limit space.

Unless we state the contrary, in what follows we assume that the bonding map is a surjection on $X$, or that its domain is restricted to the subset of $X \supset X' = \bigcap_{i \geq 0} f^i(X)$ on which $f$ is a surjection. When the bonding map $f$ is backward-moving, the corresponding forward–in–time dynamics can be described by a map acting on the inverse limit space, as follows:

$$\sigma: \varprojlim \{X, f\} \to \varprojlim \{X, f\}$$
$$\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \tag{8}$$

Notice that the map $\sigma$ is invertible, with inverse $\sigma^{-1}(x_2, x_3, \ldots) = (f(x_2), f(x_3), \ldots) = (x_1, x_2, \ldots)$. (As a matter of fact, the maps $\sigma/\sigma^{-1}$ are homeomorphisms.)

In the following discussion and in the proofs of our results, we often need to use the projection function $\pi_i, \ i = 1, 2, \ldots$:

$$\pi_i: \varprojlim \{X, f\} \to X$$
$$\pi_i(x_1, x_2, \ldots) = x_i$$

For simplicity’s sake, in what follows we put $\pi_1 = \pi$.

We can now draw some interesting conclusions, which we state here for a “simple” inverse limit space, but could be easily generalized:

- The inverse limit space $\varprojlim \{X, f\}$ is precisely the space of all the forward–in–time sequences $\hat{x} = (x_1, x_2, \ldots)$, implicitly generated by $f$, starting from initial points in $X$.

- The map $\sigma$ acting on $\varprojlim \{X, f\}$ is a (one-sided) shift map that moves a sequence one step to the left and drops the first term. Since, in the present case, $f$
is a BD map, $\sigma$ is forward-moving in time in the sense that it moves a sequence one step to the future, discarding its “oldest” element\(^7\).

- Although the dynamical system defined by $\sigma$ on the corresponding sequence space cannot be interpreted as a mathematical idealization of any real economic mechanism, as we shall see, the orbit structure of $\sigma$ reveals many interesting properties of the forward-moving orbits implicitly defined by the bonding map $f$.

The next step in our discussion is to exploit the knowledge of the (backward) dynamics of $f$ to obtain interesting information on the (forward) dynamics of $\sigma$. For this purpose, we shall recall a number of interesting results available in the mathematical literature. They are usually expressed in terms of the relations between the bonding map $f$ and its “induced homeomorphism” which, in our notation, is the map $\sigma^{-1}$. However, since the maps $\sigma/\sigma^{-1}$ are homeomorphisms, all the results that interest us here can be readily translated into analogous relations between $f$ and $\sigma$.

In order to rigorously classify the dynamical properties of a map, we focus on subsets of the state space which are persistent and dynamically indecomposable (i.e., they must be studied as a whole). Here is a basic result:

**Lemma 1.** [MR07, Lemma 1] Let $\hat{A} \subseteq \lim \{X, f\}$. Then $\hat{A}$ is closed and $\sigma$-invariant (i.e., $\sigma(\hat{A}) = \hat{A}$) if, and only if, $\hat{A} = \lim \{A, f|_A\}$, with $A = \pi(\hat{A})$ and $A$ is $f$-invariant (i.e., $f(A) = A$).

Indecomposability of a set is commonly characterized by the notion of “topological transitivity ”. Let $f : X \to X$ be a continuous map of a metric space. There exist two competing definitions of t.t., namely:

**Definition 2a.** The map $f$ is **topologically transitive (t.t.)** on $A$ provided that whenever $U$ and $V$ are open nonempty subsets of $A$ there is an integer $n$ such that $f^n(U) \cap V \neq \emptyset$.

**Definition 2b.** The map $f$ is t.t. on $A$ if there exists a point $x \in A$ such that the orbit of $x$ under $f$ is dense in $A$. 
The two definitions are equivalent if $A$ is a compact metric space with no isolated points, which is the case we consider here.

**Lemma 2.** The map $f$ is topologically transitive on $A$ if, and only if, the shift map $\sigma$ is topologically transitive on $\hat{A} = \lim \{A, f\}$.

**Proof.** Li (1992, Theorem C) proved the result for the inverse of $\sigma, \sigma^{-1}$. Since $\sigma/\sigma^{-1}$ are homeomorphisms, the extension to $\sigma$ is immediate.

In the following pages, we will discuss two main types of dynamics, namely: simple (periodic) and complex (chaotic). For periodic dynamics, we have the following, entirely intuitive result that we state without proof:

**Lemma 3.** Let $A = \{x^1, x^2, \ldots, x^n\}$ denote a periodic orbit of period $n \geq 1$ with $f(x^1) = x^2, f(x^2) = x^3, \ldots, f(x^n) = x^1$. Then, the set $\hat{A} = \lim \{A, f|A\}$ is periodic under $\sigma$ with the same period and $\pi(\hat{A}) = A^8$.

When applied to a BD map such (7), Lemma 3 simply says that the existence of periodic dynamics backward in time for the map $G$ is a necessary and sufficient condition for the existence of a periodic IEE of the same period.

In the case of chaotic dynamics, the analogous equivalence result is less obvious and, before stating it, we need some definitions and preliminary propositions.

**Definition 3.** A continuous map $f$ on a metric space $X$ is said to be chaotic on $X$ if, whenever $U$ and $V$ are open, nonempty subsets of $X$, there exist a periodic point $p \in U$ and a positive integer $k$ such that $f^k(p) \in V$, that is, every pair of open nonempty sets shares a periodic orbit.

An immediate consequence of this definition is the following corollary, which we will use in a moment.

**Corollary 1.** Let $f$ be a homeomorphism of a metric space $X$. Then $f$ is chaotic on $X$ if, and only if, its inverse $f^{-1}$ is chaotic on $X$.

The proof is straightforward: it is sufficient to interchange the two open sets $U$ and $V$ of Definition 3.

We can now state the following:
Theorem 1. Let $A$ be a metric compact set. A continuous map $f$ on $A$ is chaotic if, and only if, the map $\sigma$ is chaotic on the inverse limit space $\lim_{\rightarrow} \{A, f|_A\}$.

Proof. Our result is a straightforward consequence of Li (1992, Theorem C), who proved this statement for the inverse of $\sigma, \sigma^{-1}$, using Devaney’s definition of chaos. Given the equivalence between Devaney’s and Touhey’s definitions of chaos and Corollary 1, the required result follows immediately. See also Kennedy and Stockman (2008), where a similar result is proved.

When applied to a OLG model characterized by a BD map like (7), Theorem 1 states that the existence of chaotic dynamics for the backward moving map $G$ is a necessary and sufficient condition for the existence of chaotic dynamics for the associated forward-moving map $\sigma$ and therefore for the existence of chaotic equilibrium dynamics.

The equivalence relations proved so far are interesting but not sufficiently informative. Suppose we are studying an OLG model characterized by a BD map like $G$ of equation (7). It often happens that the (backward) iterates of a unimodal map yield many (even infinitely many) different types of orbits (e.g., periodic of many different periods, or chaotic). In this case, the results just discussed imply that there will be correspondingly many types of orbits forward-in-time, i.e., many different kinds of IEEs.

In this situation, the interesting question is: how can we distinguish between orbits that are typical and therefore interesting, and those that are exceptional and therefore negligible?

To answer this question, we take the common view that an event is “typical” (“exceptional”) if it belongs to a set that is “large” (“small”) with regards to the set of all possible events. In mathematics, there exists two main ways of “sizing up” sets: a metric approach based on the natural (Lebesgue) probability measure, and a topological approach based on the notion of (Baire) category. In our article MR07 we have established a number of results based on the topological approach. In this paper, we extend that analysis to the metric approach.
Here are some basic definitions which can be applied both to finite-dimensional state spaces usually encountered in economic models, such as subsets of $\mathbb{R}^n$, and to sequence spaces, such as inverse limit spaces. Once again, for the sake of brevity, we restrict ourselves to the essential points and for greater detail we refer the reader to MR07.

**Definition 4.** Let $f : K \to K$ be a continuous map of a metric space $K$. Let $x \in K$. Then the $\omega$-limit set of $x$ is defined to be $\omega_f(x) = \bigcap_{i \in \mathbb{Z}^+} \left( \bigcup_{m \geq i} f^m(x) \right)$. Let $A \subseteq K$ be closed and forward invariant, i.e. $f(A) = A$, then the basin of attraction of $A$ is defined to be $B(A) = \{ x \in K : \omega_f(x) \subseteq A \}$.

Broadly speaking, $\omega_f(x)$ is the set of the limit points of the $f$–orbit $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ starting from $x$; the basin of $A$ is the set of points $x$ whose $f$–orbit converges to $A$ as $n \to \infty$. Then we call $A$ an attractor provided (i) $B(A)$ is “large” and (ii) $A$ is dynamically indecomposable.

More rigorously, these two properties can defined as follows:

**Definition 5.** Let $f : K \to K$ be a continuous map of a metric space $K$. Let $A \subseteq K$ be a closed forward invariant set. Then $A$ is called a metric attractor provided: (i) $B(A)$ has positive Lebesgue measure; and (ii) $f$ is topologically transitive on $A$.

Thus, from a metric point of view, a set is “large” or “small” if it has, respectively, positive or zero measure.

Attractors, in the sense of Definition 5 are interesting because the orbits that make them up give a likely characterization of the long–run behavior of the system, whereas sets with “small” basins of attraction are made up of orbits unlikely to observe. Broadly speaking (and ignoring errors and rounding up), attractors are the objects that, transients apart, we expect to observe on the screens of our computers when we perform numerical simulations of dynamical systems, starting from randomly chosen initial conditions.

5. Basic types of backward dynamical systems arising from OLG models

In order to produce sharp results we concentrate on a class of non–invertible
maps on the interval characterized by a single critical point and called unimodal or, informally, “one–humped” maps. This class of systems concerns virtually all the one–dimensional models of backward dynamics discussed in the economic literature (e.g., OLG models of the “leisure–consumption” or the “pure exchange” type; “cash–in–advance” models). It obviously includes the benchmark OLG model discussed in Section 2. Formally, we have the following definition:

**Definition 6.** A continuous map $f$ of an interval $[a, b]$ is called (strictly) unimodal if there is a point $x^* \in (a, b)$ such that $f(x^*) \in [a, b]$ and $f$ is strictly increasing on $[a, x^*)$ and strictly decreasing on $(x^*, b]$.

Notice that if $f$ is a unimodal map, then $f$ is surjective on the interval $I = \bigcap_{n \geq 0} f^n([a, b])$. Recalling our presentation of the inverse limit space, we conclude that $\lim \{[a, b], f\} = \lim \{I, f|_I\}$. In simple words, this means that, if $f$ is a BD map as we assume here, the space $\lim \{I, f\}$ contains all the forward admissible orbits associated with it, and only them. For simplicity’s sake, and without loss of generality, whenever $f$ is a unimodal map, the factor space will be re–scaled to $I = [0, 1]$ so that the corresponding inverse limit space is $\lim \{[0, 1], f|_{[0,1]}\}$. In what follows, all the unimodal maps under consideration will be assumed to be continuous.

To prepare some of our results, we need the following preliminary definitions:

**Definition 7.** A $C^3$ unimodal map $f$ on the interval is said to be quasiquadratic (q.q.) if any sufficiently small perturbation of $f$ in the $C^3$ topology is topologically conjugate to a quadratic map.

**Definition 8.** For a $C^3$ unimodal map $f$ on the interval, the Schwarzian derivative of $f$, $Sf$ is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

If $Sf(x) < 0$ for all $x \neq x^*$, we say that $f$ is a $S$–unimodal map.
Remark 1. (i) We can look at the Schwarzian derivative as a property of the curvature of the first derivative of the function $f$. Considering that

$$\frac{d^2|f'(x)|^{-1/2}}{dx^2} = -(1/2)|f'(x)|^{-1/2}Sf(x)$$

we conclude that $Sf(x) < 0$ implies $|f'(x)|$ being concave over the monotone intervals of $f$. This assumption has been extensively used to prove certain important results concerning dynamical systems on the interval, but it is restrictive and, in general, it is not even necessary. As a matter of fact, the sign of $Sf(x)$ is not invariant w.r.t. a smooth change of co-ordinates and it is perfectly possible to have two maps $f(x), g(x)$ with $Sf(x) > 0, Sg(x) < 0$ on the respective domains, which are topologically, or even smoothly conjugate and therefore have the same dynamical properties.

(ii) A more natural class to work with is the family of quasiquadratic unimodal maps, as in definition 7, of which $S$-unimodal maps form an open subset. First, the property of being quasiquadratic – differently from having $Sf < 0$ – is preserved by smooth conjugacy. Second, in view of the fact that almost all (finitely) renormalizable unimodal maps have a quasiquadratic renormalization, certain important properties of the entire class of unimodal maps can be proved restricting the analysis to q.q. maps. Finally, the class of quasiquadratic unimodal maps defines the most general setting where bifurcations behave as for the quadratic family.\(^\text{10}\)

In what follows, we assume q.q. when it is necessary to obtain sharp results. In these cases, whenever possible, we also explain the consequences of relaxing that assumption. As we shall see, however, in the case of Theorem 3, which covers by far the most common type of BD maps arising in one-dimensional OLG models, our main result can be reached using assumptions that follow directly from the economic hypotheses of the model.

In this paper, we identify three basic subclasses of unimodal maps, labeled Type A, B and C maps. The identification of each case depends on the basic features of the controlling (backward) map $f$ and therefore on the underlying structural functions. The next step is a description of the economic characterizations of type A, B and C maps occurring in the OLG models and the corresponding mathematical definitions.
Type A maps

This type of map is exemplified by the (Samuelson) OLG models characterized by a noninvertible backward–moving offer curve with the following properties: (i) two steady states ("monetary" and "non–monetary") exist; and (ii) either there is no (second period) utility saturation, or the saturation value of the old agent’s excess demand $\zeta$, call it $Z$, is larger than $\zeta_{MAX}$ and therefore irrelevant.$^{11}$

![Figure 2. A Type A map](image)

A precise mathematical definition is the following:

**Definition 9.** A “Type A map” is a unimodal map such that $f(a) = a$ and $f(x^*) < b$.

A Type A map, restricted to $I = [0, f(x^*)]$ so that $f|_I$ is surjective and re–scaled to $[0, 1]$, is depicted by Figure 2.

Type B maps

The crucial difference between Type A and Type B maps is that for the latter there is no “non–monetary” stationary equilibrium (the origin is not a fixed point).
This case is considered, for example in Grandmont (1989) in a simplified version of his OLG “leisure–consumption” model in which only young agents work and only old agents consume. In the notation of this paper, this implies that, for all $t$, $c_t = 0; \iota_t = \bar{\iota} = 0$ and consequently $z_t = -l_t$ and $\zeta_t = \kappa_t$. This model is necessarily of the Samuelson type. Assuming that its offer curve is unimodal, there are two possibilities, depending on the properties of the second–period utility function $U$. If $\lim_{\kappa \to 0} U'(\kappa) \kappa = 0$, we have a type A map as above. If, on the contrary, $\lim_{\kappa \to 0} U'(\kappa) \kappa > 0$, the model has only one steady state equilibrium, namely the “monetary” kind. There is nothing in the economic “first principles” to suggest that either case is exceptional and should be neglected. We call the map describing the second case “Type B”.

**Figure 3. A Type B Map**

Formally, we have the following definition:

**Definition 10.** A “Type B map” is a unimodal map such that $f(a) > a$ and $f(b) = a$.

In this case, the relevant restriction is $f|_{I}$ with $I = [f^2(x^*), f(x^*)]$, the inverse limit space $\lim \{[a, b], f\}$ is equal to $\lim \{I, f|_{I}\}$.

A representation of a restricted Type B map, re-scaled to $[0, 1]$, is provided by
In the variants of the OLG model discussed so far, it was assumed that either there was no second-period utility saturation, or that saturation was irrelevant because it occurred for levels of excess demand outside the admissible interval. We complete the picture, considering a situation characterized economically by the existence of a “monetary” and a “non-monetary” stationary equilibria, both locally unstable backward in time, and by the presence of binding (second period) consumption saturation. In this case, the controlling map is called “Type C map”. Here is a formal definition:

**Definition 11.** A map $f$ on an interval $[a, b]$ is called “Type C” if: (i) $f$ is not monotone; (ii) there is a point $x^* \in (a, b)$ such that $f$ is monotone on $[a, x^*]$ and $[x^*, b]$; (iii) $f(a) = f(b) = a$ and (iv) $f(x^*) > b$. In this case, $f: [a, b] \to [a, f(x^*)]$ is surjective but its range is a superset of $[a, b]$ and therefore $f$ is not a map from $[a, b]$ to $[a, b]$.

Notice that, contrary to what happens for type A and B maps, in this case the economically relevant interval is not $[a, f(x^*)]$ (i.e. the set of values with a non-
empty counterimage under \( f \), but \([a, b] \subsetneq [a, f(x^*)]\), i.e., the (smaller) set of values over which marginal utility is nonnegative. Once again, without loss of generality, the relevant interval \([a, b]\) can be re-scaled to the \([0, 1]\), so that the saturation point is located at 1. An example of a restricted and re-scaled Type C map is provided by Figure 4.

Offer curves of Type C may occur, for example, in the pure exchange OLG models if we assume a (second-period) utility function of the quadratic type, as employed in Gale (1973, p. 168, example 3), or Benhabib and Day (1982, p. 48, example (ii)), and the “steepest parameter” is sufficiently large.

In MR07, for each of the three basic cases described above, we provided a classification of topological attractors under various more or less restrictive conditions. Unfortunately, the topological and metric notions of “size” are quite distinct and a set may be “small” in a sense and “large” in the other (and vice versa)\(^{13}\). To avoid this difficulty, it would be desirable to prove that a certain set of interest is “large” (“small”) both in a topological and in metric sense. In MR07, this could be done satisfactorily only in the special case of Type C maps. In general, when the relevant space is not a measurable subset of \( \mathbb{R}^n \) or a manifold, there are no obvious way to define a Lebesgue measure and we are left with the sole topological alternative. In this paper, we strengthen our earlier results, defining a Lebesgue-like measure on \( \lim_{\leftarrow} \{I, f\} \) for Type A and B maps as well, which is endowed with the essential properties of the Lebesgue measure. We also prove that all the topological attractors identified in our previous work are also metric attractors with respect to that measure. This implies that the forward-in-time orbits making up those attractors are generic in a measure-theoretic sense, and the same is true for the corresponding IEEs.

Since our present results on metric attractors are an extension and a generalization of the argument used in MR07 for Type C maps, we recall it briefly in the following section.

6. **A metric attractor for Type C maps**

Since in this paper the discussion of Type C maps concerns us only as an
introduction to the analysis of Type A and B maps, in this section we do not discuss
the general case but concentrate on a specific, and particularly transparent example.

Consider again the simplified “leisure-consumption” OLG model in which only
young people work and only old people consume, i.e. \( z = -l \in (-l, 0] \) and \( \zeta = \kappa \in [0, 1] \). Putting \( V(z) = z \) and \( U(\zeta) = \mu \zeta(1 - \zeta/2) \), we obtain the following backward–
moving dynamical system in the variable \( \zeta = \kappa \) (the old agent’s excess demand):

\[
\zeta_t = F_\mu(\zeta_{t+1}) = \mu \zeta_{t+1}(1 - \zeta_{t+1})
\]

where \( F_\mu \) is a type C map for \( \mu > 4 \).

The corresponding Type C map is depicted by Figure 4 above.

For our present purposes, the crucial aspect of this map is that the counterimage
of every point \( \zeta \in [0, 1] \) under \( F_\mu \) consists exactly of two points belonging to two
disjoints subsets of \([0, 1]\), lying respectively on the left and the right side of the critical
point \( \zeta^* = 1/2 \), call them \( F_{\mu,L}^{-1}(\zeta) \) (“left inverse”) and \( F_{\mu,R}^{-1}(\zeta) \) (“right inverse”), and this
is also true of the counterimage of the counterimage and so on and so forth up to any
arbitrary order\(^{14} \). Thus, each forward–in–time sequence generated by the map \( F_\mu \)
starting from any given initial point in \( \zeta \in [0, 1] \) can be coded by a unique sequence
of two symbols, say \( \{0, 1\} \).

In order to apply the machinery of ILT to a Type C map, we need to inject a
little additional structure. From now on, we refer to a generic state variable denoted
by \( x \).

Let \( X_1 = [0, 1] \), \( X_2 = X_1 \cap F_\mu^{-1}[X_1] \) and inductively define \( X_i = X_{i-1} \cap
F_\mu^{-1}[X_{i-1}] \). Also define \( f_i : X_{i+1} \rightarrow X_i \) by \( f_i = F_\mu|_{X_{i+1}} \). That is to say, in this
case the maps \( f_i \) are all identical except that each has a different domain. The

The corresponding inverse limit space \( \lim_{\leftarrow} \{X_i, f_i\} \) is the set of all forward admissible
sequences, \( (x_1, x_2, \ldots) \), permitted by the difference equation (10), with \( x_i \in [0, 1] \).

Let \( \Lambda = \bigcap_{n \geq 0} F_\mu^{-n}([0, 1]) \). This is the (Cantor) set of points \( x \in [0, 1] \) such
that \( F_\mu^n(x) \in [0, 1], \forall n \geq 0 \). Let \( \hat{\Lambda} = \lim_{\leftarrow} \{\Lambda, F_\mu|_{\Lambda}\} \). The dynamics of \( F_\mu \) on \( \Lambda \) is well–
derstood and has been thoroughly discussed in the mathematical literature (see, for
example, Katok and Hasselblatt, (1995, pp. 80–81)). In particular, it is known that \( \Lambda \)
is a $F_\mu$–invariant, repelling set and the dynamics of $F_\mu$ on $\Lambda$ are chaotic in the sense of Devaney or, equivalently, in the sense of Definition 3. Indeed, $F_\mu|_\Lambda$ is topologically conjugate to the full shift map on the space of one–side, infinite sequences of two symbols$^{15}$, which is “the most chaotic map” in the sense that its iterations can be used as a mathematical representation of an independent stochastic process such as repeated coin tossing. Notice that, although periodic orbits of $F_\mu|_\Lambda$ are uncountably many, they form a “small” set both in a topological and in a metric sense, whereas the typical orbit of $F_\mu|_\Lambda$ is chaotic.

Equipped with this information, we now turn to the analysis of the forward–moving dynamics of $\sigma$ on $\hat{\Lambda}$. First of all, from Lemma 1 and Lemma 2 we have that $\hat{\Lambda}$ is closed and $\sigma$–invariant and that $\sigma$ is topologically transitive on $\hat{\Lambda}$.

Next, let $\{0,1\}^\mathbb{N}$ denote the (Cantor) set of unilaterally infinite sequences of symbols 0 and 1, $\{(z_1,z_2\ldots) : z_i \in \{0,1\} \text{ for each } i \in \mathbb{N}\}$ and consider the standard itinerary mapping $i(x) = 0$ if and only if $x < 1/2$ and $i(x) = 1$ otherwise.

Then we have the following result:

**Lemma 4 [MR07, Lemma 3].** Define $h : \lim \{X_i, f_i\} \to [0,1] \times \{0,1\}^\mathbb{N}$ by

$$h((x_1,x_2\ldots)) = (x_1,(i(x_2),i(x_3),\ldots)).$$

Then $h$ is a homeomorphism.

Intuitively, this means that each orbit forward–in–time starting in $[0,1]$ can be uniquely determined by the initial condition $x \in [0,1]$ and an infinite sequence of two symbols, each of them corresponding, for any $x$, to a choice between the two elements of the set $F_\mu^{-1}(x)$.

An interesting consequence of this fact is that there exists a well–defined measure on the set $\{0,1\}^\mathbb{N}$ of all infinite sequences of two symbols: it is the product of the measure on the set of two elements, denoted by 0 and 1, that assigns the value 1/2 to each element. Thus, the probability assigned to the subset of $\{0,1\}^\mathbb{N}$ including all the infinite sequences having given, finite sub–sequences of $k$ elements, is $2^{-k}$. If we call this measure $\hat{\lambda}$ and $\lambda$ is the ordinary Lebesgue measure on $[0,1]$, the measurable space $(\{0,1\}^\mathbb{N}, \hat{\lambda})$ is isomorphic (modulo 0) to $([0,1], \lambda)$. The measure $\hat{\lambda}$ is
commonly called “Lebesgue (uniform) measure” on the space of symbol sequences. Then the product measure $\nu = \lambda \times \lambda$ is a meaningful Lebesgue measure on the space $[0, 1] \times \{0, 1\}^\mathbb{N}$. It is with respect to the measure $\nu$ that, in MR07, Theorem 6, we could prove that the Cantor set $\mathcal{A}$ defined above is the unique metric attractor for the shift map $\sigma$ associated with the Type C map in question (and it is also a topological attractor).

7. A Lebesgue–like measure on the inverse limit space

A quick glance at Figures 2 and 3 will suggest that for Type A or B maps we cannot establish a one–to–one correspondence between a forward–in–time orbit starting from a given initial point in $[0, 1]$ and a sequence of two symbols, for the simple reason that there exist subintervals of $[0, 1]$ over which those maps have a single inverse. Consequently we cannot use the product measure $\nu$ as defined above.

In search of a viable alternative, we start with some broad methodological considerations.

First, let us recall condition (i) of Definition 5: an attractor is the limit set of orbits originating from a set of initial conditions of positive Lebesgue measure. This condition may be expressed by saying that if we choose an initial condition randomly with regards to the uniform probability density, there is a non-zero probability that the orbit from the chosen initial condition converges to the attractor (see, Ott, 2006). As we have explained elsewhere, for a forward–moving map $F$ like (1) and (6) choosing initial conditions in the domain of $F$ (usually a subset of $\mathbb{R}^n$) is equivalent to choosing forward–in–time orbits. On the contrary, for a BD map $G$ like (2) or (7), since to each point in the domain of $G$ there may correspond many forward–in–time orbits, initial conditions must be chosen in the domain of the corresponding shift map $\sigma$, i.e, the (inverse limit) space of forward–in–time orbits.

Second, the adoption of Lebesgue measure (uniform probability density) for sizing up the set of initial conditions, which is standard in the discussions of dynamical systems in economics and everywhere else, can be seen as an application of the Bernoulli–Laplace’s Principle of Insufficient Reason (PIR)$^{17}$. In its discrete version –
which is relevant here as well as in Section 6 – the PIR requires that if there are \( n > 1 \) mutually exclusive and collectively exhaustive possibilities and if the \( n \) possibilities are similar in all discernible relevant respects, then to each possibility should be assigned a probability equal to \( 1/n \), i.e., the possibilities should have equal probability.

A modern and more sophisticated version of the Principle of Indifference is the Maximum Entropy Principle (MEP). According to the famous formula of Shannon, the information entropy function of \( n \) mutually exclusive events \( E_i, i = 1, 2, \ldots, n \), to each of which a probability \( p_i \) is assigned, is defined as

\[
H(p_1, p_2, \ldots, p_n) = -K \sum_{i=1}^{n} p_i \ln p_i,
\]

with \( K \) an arbitrary, positive constant. The principle states that “in making inferences on the basis of partial information, we must use that probability distribution which has the maximum entropy subject to whatever is known [emphasis added]”. Note that, in this case, “entropy” is a synonym of “uncertainty”. In the case of absolute ignorance about the events \( E_i \), the maximization of the information uncertainty, with the constraint that \( \sum_{i=1}^{n} p_i = 1 \), yields \( p_i = 1/n \).

In the case of Type C maps, the mutually exclusive possibilities concerned by the PIR/MEP are the two inverses of each point \( x \in [0, 1] \) under the map \( f \).

On the other hand, for Type A and B maps “what is known”, i.e., the hypotheses of the underlying models and their consequences, imply that the corresponding function \( f \) has a single inverse over a certain subset of \([0, 1]\) and a double inverse over another, without any criterion for choosing between the two alternatives. Thus, the application of the PIR/MEP principles requires that whenever there are two possibilities (inverses), we assign equal probabilities (1/2) to each of them and, when a unique possibility (inverse) exists, we assign it probability one.

What follows is a formal definition of a probability measure defined on the inverse limit space that complies with the requirements of the Principles of Insufficient Reason and Maximum Entropy while having some fundamental properties of the Lebesgue measure.
Next, we construct the required measure. Let $\hat{s} = (s_1, s_2, \ldots) \in \{0, 1\}^\mathbb{N}$. Let $f: [0, 1] \to [0, 1]$ be a Type A or Type B map and let $f_L^{-1}(x), f_R^{-1}(x)$ denote the “left” and “right inverse” of $f$, respectively.

We now define a function $h_f : [0, 1] \times \{0, 1\} \to [0, 1]$. First, if $f$ is a re-scaled Type A map, put $i \geq 1$ and define $h_f$ by

$$h_f(x_i, s_i) = \begin{cases} f_L^{-1}(x_i), & \text{if } x_i \in [0, f(1)); \\ f_L^{-1}(x_i), & \text{if } x_i \in [f(1), 1] \text{ and } s_i = 0; \\ f_R^{-1}(x_i), & \text{if } x_i \in [f(1), 1] \text{ and } s_i = 1. \\
\end{cases}$$

Next, if $f$ is a re-scaled Type B map then define $h_f$ by

$$h_f(x_i, s_i) = \begin{cases} f_R^{-1}(x_i), & \text{if } x_i \in [0, f(0)); \\ f_L^{-1}(x_i), & \text{if } x_i \in [f(0), 1] \text{ and } s_i = 0; \\ f_R^{-1}(x_i), & \text{if } x_i \in [f(0), 1] \text{ and } s_i = 1. \\
\end{cases}$$

Let $(s_1, s_2, \ldots) \in \{0, 1\}^\mathbb{N}$. Then given a point $x_i \in [0, 1]$ let $x_{i+1} = h_f(x_i, s_i)$. Define $H: [0, 1] \times \{0, 1\}^\mathbb{N} \to \lim \{[0, 1], f\}$ by

$$H(x_1, (s_1, s_2, \ldots)) = (x_1, h_f(x_1, s_1), h_f(x_2, s_2), \ldots)$$

The map $H$ is surjective but, generally, not injective.

Consider now the function:

$$\lambda = \nu \circ H^{-1}$$

where $\nu = \lambda \times \hat{\lambda}$ is defined in Section 6.

From the definition of $\nu$, and considering the fact that $\hat{\lambda}$ is the product of the measure on the set of two elements assigning the value $1/2$ to each of them, it readily follows that $\lambda$ satisfies the requirements of the PIR/MEP principles.

We now proceed to prove that $\lambda$ also possesses some basic properties of the Lebesgue measure, according to the following definition:
Definition 12. Let be: $X$ a compact metric space, $\mathcal{B}$ the Borel algebra on $X$ and $\lambda : \mathcal{B} \to \mathbb{R}$ a measure on $\mathcal{B}$. We say that $\lambda$ is Lebesgue-like provided: 1. $\lambda$ is a positive Borel measure; 2. if $U \subseteq X$ is open then $\lambda(U) > 0$; 3. if $x \in X$ then $\lambda(\{x\}) = 0$

We can state the following theorem:

Theorem 2. $\lambda$ is a Lebesgue-like measure.

Proof: See Appendix A.

But this is not all. For a class of subsets, $\mathcal{I}$, of the inverse limit space that are the analogues of subintervals of the inverse limit space, we have that the $\lambda$-measure of $J \in \mathcal{I}$ has a fundamental property that the Lebesgue measure has on subintervals of $I \subset \mathbb{R}$, namely it is translation–invariant.

To see that, let us define the set

$$\mathcal{I} = \{ \pi^{-1}(I) : I \text{ is a subinterval of } [0,1] \}.$$

Then if $J \in \mathcal{I}, J = \pi^{-1}(I)$ for some subinterval $I$ of $[0,1]$ and so we can call $J$ a subinterval of $\lim \{ [0,1], f \}$. By Lemma A.1, we see that $\lambda( J ) = \lambda( I )$ which is the length of $I$. For this reason, we can say that $\lambda( J )$ is the length of $J$.

For a subinterval $J = \pi^{-1}( I )$ we define translation by $t \in \mathbb{R}$ mod 1 by

$$J + t = \pi^{-1}( I + t( \mod 1 ) )$$

Again, by Lemma A.1, we see that

$$\lambda( J + t ) = \lambda( J ).$$

Therefore the measure $\lambda$ shares all of the significant properties of Lebesgue measure in this setting.

8. Metric attractors for the maps of Type A and B

Equipped with the results of Section 7, we are now ready to establish the existence of metric attractors for the Type A and B maps as well. Since the proofs of the following Theorems are quite technical, we relegate them to Appendix A.
First, we have the following Theorem for Type A maps:

**Theorem 3.** Let \( f : [0, 1] \to [0, 1] \) be a unimodal Type A map with \( f(1) > 0 \) and \( f'(0) > 1 \). Let 0 and \( \bar{x} \in (x^*, 1] \) be the only fixed points of \( f \). Let \( \hat{0} = (0, 0, 0, \ldots) \in \lim \{[0, 1], f\} \). Then \( \{0\} \) is the only metric attractor for \( \lim \{[0, 1], f\} \) under \( \sigma \).

A rigorous proof of Theorem 3 can be found in Appendix A. In words, this theorem means that, if a OLG model is characterized by a unimodal map of Type A with the specifications of the Theorem, \( \lambda \)–almost all orbits forward–in–time converge asymptotically to the “non–monetary” stationary state. This is indeed the case for the Grandmont–like basic model discussed in Section 2.

**Remark 2.** For the class of OLG models described by Type A maps, the standard economic assumptions are sufficient to guarantee that \( f'(0) > 1 \) and that there exist two stationary equilibrium states, one positive (“monetary”) and a second one (“non–monetary”), located at the origin (cf. footnotes 3 and 4). We have excluded the case in which the “monetary” fixed point is located to the left of the critical point (or on it), because in this case the restricted map is monotonically increasing and its dynamics are trivial: all the admissible forward–in–time–orbits converge to the “non–monetary” fixed point, except one, namely \( \hat{x} = (\bar{x}, \bar{x}, \ldots) \). For more general applications of Type A unimodal maps, the assumption that \( f \) is q.q. (or \( Sf < 0 \)) is sufficient (though not necessary) to guarantee the uniqueness of the metric attractor \( \hat{0} \) for the map \( \sigma \).

For maps of Type B in which the unique fixed point is located in the subinterval \((x^*, 1)\)\(^{19}\), there are two very different situations. Suppose \( q \) is the least point in \([x^*, 1] \) fixed under \( f^2 = f \circ f \). Then the first, simpler case occurs when \( f(0) > q \). We have the following result:

**Theorem 4.** Let \( f : [0, 1] \to [0, 1] \) be a quasiquadratic unimodal Type B map with a unique fixed point \( \bar{x} \in (x^*, 1) \), and let \( |f'(\bar{x})| > 1 \) and \( f(0) > q \). Then \( \hat{x} = (\bar{x}, \bar{x}, \ldots) \) is the only metric attractor for \( \lim \{[0, 1], f\} \) under \( \sigma \).

**Proof:** see Appendix A.
Remark 3. As we mention in the proof, if the map $f$ of Theorem 4 is q.q., $q = \bar{x}$. If $f$ is not q.q., this need not be true and the situation is more complicated. Specifically, the shift map $\sigma$ associated with $f$ may have multiple attractors. Two main possibilities arise: if $f(0) > f(q)$, depending on the initial conditions, the forward–in–time orbits will be attracted to $\bar{x}$ or to one of the many possible period–two cycles, unstable under $f$ and located in the interval $[q, f(q)]$. If $q \leq f(0) \leq f(q)$, for appropriate initial conditions, there may also exist a forward–in–time chaotic attractor, formed by two subsets of the ”horseshoe” type, each visited by the iterates of $\sigma$ with periodicity $2^{20}$.

Next we consider the more interesting and complicated case of $f(0) < q$, where, we recall, $q$ is the first (smallest) fixed point of $f^2$ in the interval $[x^*, 1]$. This corresponds to a “chaotic region” of the parameter space, in the sense that the following properties hold (cf. Barge and Diamond, 1994 and Ingram, 2000b): (i) $\lim \{[0, 1], f\}$ is indecomposable; (ii) there exists a $f$–invariant chaotic set (a horseshoe); (iii) the maps $f$ on $[0, 1]$ and $\sigma$ on $\lim \{[0, 1], f\}$ have positive topological entropy. We prove below that, in this case, if $f$ satisfies some fairly general assumptions and has an attractor $P \neq [0, 1]$ that is either periodic or chaotic, then there is an invariant Cantor set, $\Lambda$, which generates a metric attractor for $\sigma$, $\Lambda = \lim \{\Lambda, f_{\Lambda}\} \subset \lim \{[0, 1], f\}$.

To prepare a rigorous statement of our result, we start with a general result, proved with some variations in a series of recent papers21 and stating that, for a family of real analytic, unimodal maps, for Lebesgue–almost all parameters, the maps are either “regular” or “stochastic” (chaotic)22. Here a map $f$ is called “regular” if it is hyperbolic, has a non–degenerate critical point, $x^*$, i.e. $f''(x^*) \neq 0$, which is not periodic or preperiodic. Regular maps have one or more periodic attractors, one of them containing $x^*$ in its immediate basin; stochastic maps have a unique chaotic attractor (transitive cycle of intervals, supporting an absolutely continuous invariant measure), plus perhaps a finite number of periodic attractors, $x^*$ being in the basin of the chaotic attractor. Accordingly, parameters generating attractors of the solenoid type (roughly speaking, attractors that are aperiodic but not chaotic) form sets of
measure zero in the parameter space. If, in addition, the map \( f \) is quasiquadratic (q.q.) in the sense of Definition 7, it has either a unique periodic attractor or a unique chaotic one, each of them having \( x^* \) in its basin (see Avila and Co., quoted above).

Next, we make use of a result of W.T. Ingram (1995) that we restate using our notation:

**Theorem [Ingram (1995, Th.6)].** Let \( f \) be a Type B unimodal map on \([0,1]\), with critical point \( x^* \) and a unique fixed point \( \bar{x} \in (x^*,1) \). Then \( f \) has a periodic point with odd period greater than 1 if, and only if, \( f(0) < q \) where \( q \) is the first fixed point of \( f^2 \) that is in the interval \([x^*,1]\).

**Lemma 5.** Let \( f \) be a type B unimodal q.q. map with \( f(0) < q \). Then \( f \) has infinitely many periodic orbits.

**Proof:** The result follows immediately from Ingram’s 1995 theorem and the well-known Sarkovskii theorem.

From the stated results, we conclude that a typical quasiquadratic unimodal map \( f \) of Type B with a unique fixed point \( \bar{x} \in (x^*,1) \) and such that \( f(0) < q \) satisfies the following properties:

1. \( f \) has a unique attractor (either periodic or chaotic) \( P \neq [0,1] \);
2. there are finitely many proper subintervals \( B_0, \ldots, B_{n-1} \) (the immediate basin of \( P \)) such that:
   
   (a) \( P \subseteq \bigcup_{i=0}^{n-1} B_i \);
   
   (b) \( f(B_{i-1}) \subseteq B_i \) for \( 1 \leq i \leq n-1 \), \( f(B_{n-1}) \subseteq B_0 \);
   
   (c) \( \partial B_i \) is not in the basin of \( P \);
   
   (d) \( \partial B_i \cap \partial B_j = \emptyset \) for all \( i \neq j \).

Notice that parts (a), (b) and (c) of assumption (2) follow from the definition of the attractor as either periodic (and hence having an immediate basin of subintervals which are cyclically permuted) or as chaotic (a transitive cycle of intervals.). Lemma
5 establishes that if \( f \) is q.q. and \( f(0) < q \), then \( f \) has infinitely many periodic points. This implies that infinitely many distinct orbits are not attracted to \( P \) and hence there must be intervals in the complement of the immediate basin. Assumption 2(d) will then follow.

Let
\[
B = \bigcup_{i=0}^{n-1} B_i
\]
and
\[
\Lambda = \{x \in [0,1] : f^n(x) \notin B \text{ for all } n \in \mathbb{N} \} = [0,1] \setminus \bigcup_{n \in \mathbb{N}} f^{-n}(B).
\]

Since each \( B_i \) is a subinterval that does not contain its boundary, we see that each \( B_i \) is an open interval. Thus \( B \) is an open set and \( [0,1] \setminus B \) contains nondegenerate intervals (by assumption 2(d)).

**Lemma 6.** \( \Lambda \) is an invariant Cantor set.

**Proof:** see Appendix A.

The intervals \( B_i \) are cyclically permuted by \( f \) and only one of them can contain the critical point \( x^* \). This implies that there is a word in 0 and 1, \( W \), such that \( x \in \Lambda \) if and only if the itinerary of \( x \) does not contain \( W \) or \( x^* \). Since \( \Lambda \) contains none of the preimages of \( x^* \) we see that the itinerary mapping, \( i \), is a homeomorphism from \( \Lambda \) onto \( i(\Lambda)^{23} \). Since the points in \( \Lambda \) are characterized by the property of their itinerary not containing the word \( W \) we see that \( i(\Lambda) \) is a subshift of finite type, and hence \( \Lambda \) is conjugate to a subshift of finite type. By Theorem 1 we see then that \( \sigma \) acting on the inverse limit of \( \Lambda \), which we denote by \( \hat{\Lambda} = \lim_{\to} \{\Lambda, f|_\Lambda\} \), is chaotic. In particular it is topologically transitive on \( \hat{\Lambda} \).

We have then the following Theorem:

**Theorem 5.** Let \( f \) be a Type B unimodal map of \([0,1]\) with fixed point \( x^* \in (x^*, 1) \) and such that \( f(0) < q \). Suppose that \( f \) satisfies assumptions (1) and (2) above. Let
\[ \Lambda = \{ x \in [0, 1] : f^n(x) \notin B \text{ for all } n \in \mathbb{N} \}, \text{ and let } \hat{\Lambda} = \{ \hat{x} \in \lim \{ [0, 1], f \} : x_i \in \Lambda \text{ for all } i \in \mathbb{N} \}. \text{ Then } \hat{\Lambda} \text{ is the metric attractor for } \sigma \text{ in } \lim \{ [0, 1], f \}. \]

PROOF: see Appendix A.

We now complete the picture discussing the special case of Type B maps in which the metric attractor of \( f \) is the entire interval, i.e., \( P = [0, 1] \), and finding out the corresponding metric attractor for the shift map \( \sigma \).

Before stating our main result, we need a preliminary Lemma.

**Lemma 7.** Let \( f \) be a Type B map with \( 0 < f(0) < \bar{x} \), with an attractor \( P = [0, 1] \). Then \( f \) is topologically exact on \([0, 1]\).

PROOF: see Appendix A.

We can now state the following theorem:

**Theorem 6.** Let \( f \) be a Type B map with \( 0 < f(0) < \bar{x} \) and a metric attractor \( P = [0, 1] \). Then the only metric attractor for the corresponding map \( \sigma \) on the space \( \lim \{ [0, 1], f \} \) is the entire space.

PROOF: see Appendix A.

9. Conclusions

From the formal results proved in the previous Sections, we can draw a number of interesting, and sometimes puzzling conclusions concerning the intertemporal economic equilibria (IEEs) generated by overlapping generations models described mathematically by a BD map \( f \) on the interval \([0, 1]\) and the corresponding shift map \( \sigma \) on the associated inverse limit space \( \lim \{ [0, 1], f \} \).

1. If the BD map \( f \) is of Type A (no utility saturation, two stationary equilibria, “monetary” and “non–monetary”), no matter what the backward–in–time dynamics of \( f \) are, the typical IEE is characterized by an orbit converging forward–in–time
to $\hat{x} = (\bar{x}, \bar{x}, \ldots)$, i.e. an infinite repetition of the “non–monetary” stationary equilibrium. This result provides a more general and rigorous proof for certain earlier propositions (see, Gale, *op.cit.*, p. 166, Th. 4), asserting attractiveness forward–in–time of the “non–monetary” steady state equilibrium only on the basis of its local instability backward in time. The discussion of Type B and C maps shows that, contrary to a commonly held view, instability backward in time of a stationary (or a periodic) equilibrium is not a sufficient condition for that equilibrium to be an attractor forward–in–time *globally*, i.e., when we consider all the forward admissible orbits.

2. For Type B maps (no utility saturation, a unique, “monetary” stationary equilibrium), two main situations may occur, with some variations therein.

   (i) The simpler, “periodic case” occurs if the single fixed point of $f$ is unstable and $f(0) > q$, that is, broadly speaking, if there exists a periodic orbit of period 1 or 2 in the subinterval in which the map $f$ has a single inverse. In this case, if $f$ is q.q., the typical IEE is characterized by orbits forward–in–time converging to the unique stationary equilibrium $\bar{x}$. If the hypothesis of q.q. is dropped, there may be IEEs characterized by (one or more) periodic attractors of period two, plus possibly by “periodic chaos” of period two.

   (ii) The structure of forward–in–time orbits is more complicated if $f(0) < q$ – the “chaotic case”. Here, if $f$ is q.q. and has a unique metric attractor $P \subset [0, 1]$ which is periodic or chaotic, the typical IEE is characterized by orbits forward–in–time converging to a unique metric attractor, a Cantor set on which the dynamics are chaotic.

3. Finally, in the special case in which $P = [0, 1]$ is the unique metric attractor for the BD map $f$, the typical IEE is characterized by chaotic, forward–in–time orbits which are dense on the entire interval.

**Appendix A**
Proof of Theorem 2.

Let $\mathcal{M}$ be the $\sigma$-algebra of Borel sets in $[0,1]$. Let $\mathcal{N}$ be the $\sigma$-algebra of measurable sets of $\{0,1\}^\mathbb{N}$. Let $\mathcal{A}$ be the algebra of finite, disjoint unions of rectangles in $[0,1] \times \{0,1\}^\mathbb{N}$ such that if $A \in \mathcal{A}$ and $A = \bigcup_{i=1}^n A_i$ then each $A_i = B_i \times C_i$ where $B_i \in \mathcal{M}$ and $C_i \in \mathcal{N}$. Often $\mathcal{A}$ is denoted by $\mathcal{M} \otimes \mathcal{N}$. Let $A \in \mathcal{A}$ with $A = \bigcup_{i=1}^n A_i$, the $A_i$s disjoint, and each $A_i = B_i \times C_i$ where $B_i \in \mathcal{M}$ and $C_i \in \mathcal{N}$. Then the measure of $A$ is defined by

$$\nu(A) = \sum_{i=1}^n \lambda(B_i) \cdot \hat{\lambda}(C_i)$$

It is a standard result that the algebra $\mathcal{A}$ generates a $\sigma$-algebra $\mathcal{B}$ and that $\nu$ induces a measure (which we also denote by $\nu$) on $\mathcal{B}$.

The set function $\lambda$ is defined on every set $\hat{K} \subseteq \lim_\leftarrow \{[0,1], f\}$ with the property that $H^{-1}(\hat{K}) \in \mathcal{B}$. Let $\mathcal{S}$ be the collection of all such sets, i.e. $\mathcal{S}$ is the set of all $\hat{K}$ with the property that $H^{-1}(\hat{K}) \in \mathcal{B}$. We wish to show that $\lambda$ is a Borel measure. To that end define $\mathcal{R} = \{\hat{M} \subseteq \lim_\leftarrow \{[0,1], f\} : H^{-1}(\hat{M}) \in \mathcal{A}\}$. It is clear that $\mathcal{R} \subseteq \mathcal{S}$. We will show that $\mathcal{R}$ contains all of the open sets in $\lim_\leftarrow \{[0,1], f\}$, and that it is an algebra of sets. We will also show that $\mathcal{S}$ is a $\sigma$-algebra of sets, and therefore it will contain the Borel $\sigma$-algebra.

Lemma A.1. Let $K \subseteq [0,1]$ be Lebesgue measurable. Then $\pi^{-1}(K) \in \mathcal{R}$ and $\pi^{-1}(K)) = \lambda(K)$.

Proof. Let $\hat{K} = \pi^{-1}(K)$. We will show that for every $(s_1, \ldots) \in \{0,1\}^\mathbb{N}$ and for every $x_1 \in K$ there is a point $\hat{x} \in H^{-1}(\pi^{-1}(K))$ with $\hat{x} = (x_1, (s_1, s_2, \ldots))$.

Let $x_1 \in K$. Let $(s_1, s_2, \ldots) \in \{0,1\}^\mathbb{N}$. By definition, $f^{-1}(x_1) = f^{-1}_L(x_1) \cup f^{-1}_R(x_1)$, and so every point, $\hat{z}$, in $\pi^{-1}(x_1) \subseteq \hat{K}$ has the property that $z_1 = x_1$ and $z_2 \in f^{-1}_L(x_1) \cup f^{-1}_R(x_1)$. If $x_1$ is such that both $f^{-1}_L(x_1)$ and $f^{-1}_R(x_1)$ are defined then clearly there is a point $x_2 \in f^{-1}(x_1)$ such that $x_2 \in f^{-1}_L(x_1)$ or $x_2 \in f^{-1}_R(x_1)$ depending on the value of $s_1$. If instead though one of $f^{-1}_L$ or $f^{-1}_R$ is not defined for
Let $x_1$ then each $\tilde{z} \in \pi^{-1}(x_1)$ is of the form $\tilde{z} = (x_1, x_2, z_3, z_4 \ldots)$ where $x_2$ is the unique inverse image of $x_1$ under $f$. In this case $h_f(x_1, 0) = x_2$ and $h_f(x_1, 1) = x_2$. Thus we see there are points $(x_1, (0, \ldots))$ and $(x_1, (1, \ldots))$ in $H^{-1}(\hat{K})$. So in this case the specific value of $s_1$ is not important.

Continuing, suppose that $x_j$ has been chosen for all $1 \leq j < n$ to be compatible with the word $s_1, s_2 \ldots s_{n-1}$. Again we see that $f^{-1}(x_{n-1}) = f_L^{-1}(x_{n-1}) \cup f_R^{-1}(x_{n-1})$, and if both are defined then we can choose $x_n$ to be in $f_L^{-1}(x_{n-1})$ if $s_n = 0$ and $x_n \in f_R^{-1}(x_{n-1})$ if $s_n = 1$. If only one is defined then the value of $s_n$ does not matter and $x_n$ is uniquely defined. This leads to a point $\hat{x} = (x_1, x_2, \ldots) \in \pi^{-1}(x_1) \subseteq \pi^{-1}(K) = \hat{K}$ with the property that $H^{-1}(\hat{x}) \ni (x_1, (s_1, s_2, \ldots))$. It follows that $H^{-1}(\hat{K}) = K \times \{0, 1\}^N$. Thus $\lambda(\hat{K}) = \lambda(K)$.

We shift our focus to verifying that $\lambda$ is a measure in the case that the bonding map is unimodal of Type B. The case that it is unimodal of Type A is similar.

**Lemma A.2.** Let $f$ be a unimodal Type B map. Let $n \in \mathbb{N}$. Let $K \subseteq [0, 1]$ be Lebesgue measurable such that

1. for every $m \leq n$, $f^m(K) \subseteq [0, x^*)$ or $f^m(K) \subseteq (x^*, 1]$, and
2. if $f^m(K) \cap [0, f(0)) \neq \emptyset$ then $f^m(K) \subseteq [0, f(0))$.

Then

$$H^{-1}(\pi_n^{-1}(K)) \in \mathcal{R}$$

and for each $i \leq n$ there is a nonempty set $L_i \subseteq \{0, 1\}$ such that

$$H^{-1}(\pi_n^{-1}(K)) = f^n(K) \times \prod_{i=1}^{n} L_i \times \{0, 1\}^N$$

Moreover

$$\lambda(\pi_n^{-1}(K)) = \lambda(f^n(K)) \cdot \prod_{i=1}^{n} \frac{|L_i|}{2^n}$$

and $|L_i| = 1$ if $f^{n-i}(K) \cap [0, f(0)) = \emptyset$ while $|L_i| = 2$ if $f^{n-i}(K) \subseteq [0, f(0))$.

**Proof.** Let $\hat{K} = \pi_n^{-1}(K)$, and let $\hat{x} = (x_1, x_2 \ldots) \in \hat{K}$. Then $x_n \in K$ and for every $m < n$, $f^m(x_n) = x_{n-m} \in f^m(K)$. So $x_1 \in f^{n-1}(K)$ and $x_2 \in f^{n-2}(K)$. By
assumption (1), either \( f^{n-2}(K) \subseteq [0, x^*) \) or \( f^{n-2}(K) \subseteq (x^*, 1] \). This implies that \( x_2 \) is either equal to \( f^{-1}_R(x_1) \) or \( f^{-1}_L(x_1) \). If \( f^{-1}(K) \cap [0, f(0)) = \emptyset \) then there is a unique \( s_1 \in \{0, 1\} \) such that every point in \( H^{-1}(\hat{K}) \) with first coordinate \( x_1 \) has some \((s_1, \ldots)\) as its second coordinate. In this case \( L_1 = \{s_1\} \). If instead \( f^{-1}(K) \subseteq [0, f(0)) \) then we see that \( L_1 = \{0, 1\} \) because both \( h_f(x_1, 0) \) and \( h_f(x_1, 1) \) equal \( x_1 \).

Continuing recursively, suppose that \( L_m \) is defined as above and consider \( x_m \in f^{n-m}(K) \). Then \( x_{m+1} \in f^{-1}(x_m) \) and by assumption (1), this is either in \([0, x^*)\) or \((x^*, 1]\). Either way there is a unique choice of \( f^{-1}_L(x_m) \) or \( f^{-1}_R(x_m) \). Again, if \( f^{n-m}(K) \cap [0, f(0)) = \emptyset \) then there is a unique \( s_m \in \{0, 1\} \) that corresponds with the choice of \( f^{-1}_L(x_m) \) or \( f^{-1}_R(x_m) \). In this case we take \( L_m = \{s_m\} \). In the other case, \( f^{n-m}(K) \subseteq [0, f(0)) \), we see that \( L_m = \{0, 1\} \) because by definition, both \( h_f(x_m, 0) \) and \( h_f(x_m, 1) \) equal \( x_{m+1} \) in this case.

So for each \( m < n \) we have defined a set \( L_{m+1} \subseteq \{0, 1\} \) that encodes the choice of preimage of a point \( x_m \in f^{n-m}(K) \) consistent with the bonding map. Notice that the specific point \( \hat{x} \) did not influence our choice of \( L_m \). In fact the set \( L_m \) depends only upon

1. whether \( f^{n-m+1}(K) \subseteq [0, x^*) \) or \( f^{n-m+1}(K) \subseteq (x^*, 1] \) and
2. whether \( f^{n-m}(K) \cap [0, f(0)) = \emptyset \) or \( f^{n-m}(K) \subseteq [0, f(0)) \).

This gives us

\[
H^{-1}(\hat{K}) = f^n(K) \times \prod_{i=1}^{n} L_i \times \cdots
\]

The fact that the final factor in \( H^{-1}(\hat{K}) \) is \( \{0, 1\}^N \) follows in a similar fashion as in the proof of the previous lemma where instead of starting with \( x_1 \in K \) we start with \( x_n \in K \).

**Theorem A.3.** Let \( f \) be a Type B unimodal map. Let \( n \in \mathbb{N} \) and let \( K \subseteq [0, 1] \) be measurable. Then \( \pi_n^{-1}(K) \in \mathcal{R} \).

**Proof.** Let \( \hat{K} = \pi_n^{-1}(K) \). We show that \( H^{-1}(\hat{K}) \) is a finite union of measurable rectangles in \([0, 1] \times \{0, 1\}^N \).
Let \( P_n = \{0 = x_1 < x_2 < \cdots < x_k = 1\} \) be a partition of \([0, 1]\) such that

\[
P_n = \left( \bigcup_{m=0}^{n} f^{-m}(x^*) \right) \cup \left( \bigcup_{i=0}^{n} f^{-i}(f(0)) \right)
\]

Let \( K = \bigcup_{j=1}^{p} K_j \) be the decomposition of \( K \) induced by \( P_n \). Then each \( K_j \) has the following properties

1. either \( f^m(K_j) \subseteq [0, x^*) \) or \( f^m(K_j) \subseteq (x^*, 1] \), and
2. either \( f^m(K_j) \cap [0, f(0)) = \emptyset \) or \( f^m(K_j) \subseteq [0, f(0)) \). So each \( K_j \) satisfies the previous lemma. Since \( \hat{K} = \pi_n^{-1}(K) = \bigcup_{j=1}^{p} \pi_n^{-1}(K_j) \) and the \( \pi_n^{-1}(K_j)s \) are disjoint, the result follows.

**Corollary A.4.** The algebra of sets, \( \mathcal{R} \), contains all of the open sets. Thus \( S \) contains the Borel \( \sigma \)-algebra and \( \lambda \) is a Borel measure.

**Proof.** It is well known that a basis for the topology of \( \lim \{[0, 1], f\} \) is the collection \( \{\pi_n^{-1}(U)|n \in \mathbb{N} \text{ and } U \text{ an open set}\} \).

**Theorem A.5.** Let \( f \) be a unimodal Type B map. Suppose that for each open set \( U \) in \([0, 1]\), \( f^k(U) \) has positive Lebesgue measure. Then \( \lambda \) is a Lebesgue-like measure.

**Proof.** By the previous theorem we see that \( \lambda(\pi_n^{-1}(U)) > 0 \) for each open set \( U \) in \([0, 1]\).

**Lemma A.6.** Let \( Z \in S \). For each \( i \in \mathbb{N} \) let \( L_i \subseteq \{0, 1\} \) be defined by \( s_i \in L_i \) if, and only if, there is a point \( (x_1, (s_1, s_2, \ldots s_i, \ldots)) \in H^{-1}(Z) \). Then

\[
\lambda(Z) \leq \lambda(\pi(Z)) \cdot \prod_{i=1}^{\infty} \frac{|L_i|}{2^i}
\]

**Proof.** For each \( n \in \mathbb{N} \), \( H^{-1}(Z) \subseteq \pi(Z) \times \prod_{i=1}^{n} L_i \times \{0, 1\}^{\mathbb{N}} \).

This lemma gives us the following two useful facts:

1. Let \( Z \in S \). If \( \pi(Z) \) has zero Lebesgue measure then \( \lambda(Z) = 0 \).
(2) Let \( Z \in S \). If for infinitely many \( i \in \mathbb{N} \) the choice of \( s_i \) is unique and \( \pi_i(Z) \cap [0, f(0)) = \emptyset \) then \( \lambda(Z) = 0 \).

Hence \( \lambda \) is a Lebesgue–like measure.

**Proof of Theorem 3**

Let \( R = \{ \hat{x} \in \lim_{\rightarrow} \{ [0,1], f \} : \text{there exists some } N \in \mathbb{N} \text{ such that } x_n \in [0, f^2(x^*)) \text{ for all } n \geq N \} \). Since, in the case we are considering, \( f(x) > x \) for \( x \in (0, x^*) \), it follows that \( x_n \to 0 \) as \( n \to \infty \). Thus for each \( \hat{x} \in R \), \( \sigma^n(\hat{x}) \to 0 \) as \( n \to \infty \). This implies that \( R \subseteq B(\{ 0 \}) \). By the definition of \( R \), we see that \( R \supseteq \pi^{-1}[0, f^2(x^*)) \), so \( \lambda[R] > 0 \). Since \( 0 \) is a single point it cannot have a proper subset which is an attractor. This \( \{ 0 \} \) is a metric attractor for \( \lim_{\rightarrow} \{ [0,1], f \} \) and \( \sigma \).

We next show that \( \lambda[R] = 1 \). It will then follow that \( \{ 0 \} \) is the only metric attractor for \( \lim_{\rightarrow} \{ [0,1], f \} \) and \( \sigma \). Let \( K = \lim_{\rightarrow} \{ [0,1], f \} \setminus R \) and pick \( M \in \mathbb{N} \) large enough so that \( f^M_0 [0,1] \subseteq [0, f^2(x^*)) \). Then if \( \hat{x} \in K \), \( \sigma^n(\hat{x}) \in K \) for all \( n \). Thus if \( (x_1, s_1, s_2, \ldots) \in [0,1] \times \{ 0,1 \}^\mathbb{N} \) with \( H(x_1,(s_1, s_2, \ldots)) = \hat{x} \) then \( (s_1, s_2, \ldots) \) cannot have \( M \) adjacent 0’s. It is not hard to see then that no point of \( \Gamma[K] \) has a dense \( \sigma \)-orbit. By Walters, (1982, Theorem 1.7) the set of points with a dense orbit, \( D \), in this space has \( \hat{\lambda}(D) = 1 \). It follows then that \( \hat{\lambda}(\Gamma[K]) = 0 \). Q.E.D.

**Proof of Theorem 4**

First of all, if \( f \) is a q.q. map, its second iterate \( f^2 \) can have at most 3 fixed points, namely: the unique fixed point of \( f \), \( \bar{x} \), plus, possibly, two fixed points making up a cycle of period two for \( f \). Secondly, in this paper we are concerned with the case in which the restricted map \( f|_{[f^2(x^*), f(x^*)]} \) is unimodal, i.e., when \( f^2(x^*) < x^* \). If we now consider the graph of \( f^2 \) on \( [f^2(x^*), f(x^*)] \) (which, as usual, we re–scale to \( [0,1] \)), we observe that, if the period–two cycle exists, its smaller element must lie in the sub–interval \( (0, x^*) \) and the larger one in the sub–interval \( (\bar{x}, 1) \). Thus, in our case, \( q = f(q) = \bar{x} \).

Moreover, if \( f(0) > q = \bar{x} \), each of the intervals \([0, \bar{x}] \) and \([\bar{x}, 1] \) is mapped
onto itself by the second iterate of \( f, f^2 \), and therefore \( \lim_{\infty} \{[0,1], f^2\} = \lim_{\infty} \{[0, \bar{x}], f^2_{[0, \bar{x}]}\} \cup \lim_{\infty} \{[\bar{x}, 1], f^2_{[\bar{x}, 1]}\} \) (cf. Ingram (1995), considering that, due to re-scaling, the author’s points \( a, b, c, p \) correspond to our \( 0, 1, x^*, \bar{x} \), and his “Type (1) map” corresponds to our “Type B map”, respectively).

Consider now that \( F_1 = f^2|_{[\bar{x}, 1]} \) is a Type A map and \( F_2 = f^2|[0, \bar{x}] \) is topologically conjugate to a Type A map \( G = h \circ F_2 \circ h^{-1} \), with \( h(x) = h^{-1}(x) = \bar{x} - x \). Using now the same argument as in the proof of Theorem 3 (and considering that \( h^{-1}(0) = \bar{x} \)), we can establish that the shift map \( \sigma \) associated with each of the two maps \( F_1, F_2 \) has a unique metric attractor \( \hat{x} = (\bar{x}, \bar{x}, \ldots) \).

At this point, we invoke the well-known fact that \( \lim_{\infty} \{X, f\} \) is homeomorphic to \( \{X, f^n\} \) for any \( n \geq 1 \) (see, Ingram, 2000, Corollary 1.71), and the result is proved.

**Proof of Lemma 6**

Notice that since \( f \) is continuous, each \( f^{-n}(B) \) is open, and so \( \Lambda = [0,1] \setminus \bigcup_{n \in \mathbb{N}} f^{-n}(B) \) is closed and hence compact. Since the attractor \( P \) attracts Lebesgue almost every point in \([0,1]\) we see that \( \Lambda \) contains no intervals, and is therefore totally disconnected. To see that \( \Lambda \) is perfect, let \( x \in \Lambda \) and \( \epsilon > 0 \). Suppose that every \( z \in B_\epsilon(x) \) with \( z \neq x \) is in the basin of attraction of \( P \). Let \( a = x - \epsilon \) and \( b = x + \epsilon \). Each \( z \in (a, x) \cup (x, b) \) is mapped into \( B \) by some iterate of \( f \). This implies that for each such \( z \) there is a \( B_i \) and an integer \( n \) such that \( z \in f^{-n}(B_i) \). Consider \((a, x)\). If there are two different \( n, m \in \mathbb{N} \) and \( i, j \) with \( f^{-n}(B_i) \cap (a, x) \neq \emptyset \) and \( f^{-m}(B_j) \cap (a, x) \neq \emptyset \) then since \( B_i \cap B_j = \emptyset \), there must be some point \( y \in (a, x) \) that is in the boundary of one of \( f^{-n}(B_j) \) or \( f^{-m}(B_i) \). This point \( y \) is then mapped to the boundary of \( B_i \) or \( B_j \) by \( f^n \) or \( f^m \). By assumption 2(c) this point is not in the basin of \( P \). Hence we see that there cannot be two different \( n, m \in \mathbb{N} \) and \( i, j \) with \( f^{-n}(B_i) \cap (a, x) \neq \emptyset \) and \( f^{-m}(B_j) \cap (a, x) \neq \emptyset \). This implies that there is a single \( n \in \mathbb{N} \) and \( i \) such that every point in \((a, x)\) is in \( f^{-n}(B_i) \). Hence \( x \) is in the boundary of \( f^{-n}(B_i) \). By a similar argument for \((x, b)\) we find some \( m \) and \( j \) such that \( x \) is in the boundary of \( f^{-m}(B_j) \). Suppose, without loss of generality,
that $m \geq n$. Then $f^m(x)$ is in the boundary of $B_j$ and $f^m(x)$ is in the boundary of $f^m(f^{-n}(B_i)) = f^{m-n}(B_i) \subseteq B_k$ for some $k$. If $k \neq j$ then $f^m(x)$ is in the intersection $\partial B_k \cap \partial B_j$ which contradicts assumption 2(d). Thus $k = j$. This though implies that every point in $(a, b)$ is also in $f^{-m}(B_j)$. So $f^m|_{(a, b)}$ is not monotone and in fact it ‘folds’ at $x$. Thus $x$ is a preimage of $x^*$. This is a contradiction because $x^* \in B(P)$ and $x \notin B(P)$.

Proof of Theorem 5

Before we prove the theorem, we give a few lemmata. Let

$\hat{B} = \{ \hat{x} \in \lim \{ [0, 1], f \} : x_i \in B \text{ for all } i \in \mathbb{N} \}$

Lemma A.7. Let $\hat{x} \in \lim \{ [0, 1], f \}$ such that $\hat{x} \notin \hat{B}$, then $\hat{x} \in B(\hat{\Lambda})$.

Proof. To show that $\hat{x} \in B(\hat{\Lambda})$ we show that $\omega(\hat{x}) \subseteq \hat{\Lambda}$. Let $\hat{z} \in \omega(\hat{x})$, and suppose that $\hat{z} \notin \hat{\Lambda}$. Then by the definition of $\hat{\Lambda}$, there is some integer $N$ such that $z_N \notin \Lambda$. Since $z_N \notin \Lambda$ there is some $p \in \mathbb{N}$ such that $f^p(z_N) \in B$. This implies that $z_{p+N} \in B$. Since $\hat{z} \in \omega(\hat{x})$ there is an increasing sequence of integers, $m_i \rightarrow \infty$, such that $\sigma^{m_i}(\hat{x}) \rightarrow \hat{z}$. Since the coordinate maps are continuous we see that $\pi_N(\sigma^{m_i}(\hat{x})) \rightarrow \pi_N(\hat{z})$ as $i \rightarrow \infty$. This implies that $x_{N+m_i} \rightarrow z_N$ as $i \rightarrow \infty$ and since $f$ is continuous we see that $x_{N+m_i+p} \rightarrow z_{N+p} \in B$ as $i \rightarrow \infty$. Recall that $B$ is an open set. Hence there is some $R \in \mathbb{N}$ such that $x_{N+m_i+p} \in B$ for all $r \geq R$.

We next show that this implies that $\hat{x} \in \hat{B}$ which will lead to our contradiction. We show this by showing that $x_i \in B$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ and choose $r \geq R$ so large that $N + m_r + p \geq i$. Then by above and our choice of $r$ we see that $x_{N+m_r+p} \in B$. Since $f$ maps $B$ into $B$ we see that $x_i = f^{N+m_r+p-i}(x_{N+m_r+p}) \in f^{N+m_r+p-i}(B) \subseteq B$

Hence $\hat{x} \in \hat{B}$, a contradiction.

It follows from the lemma and the definition of $\hat{B}$ that

$B(\hat{\Lambda}) = \{ \hat{x} : x_j \notin B \text{ for some } j \in \mathbb{N} \} = \lim \{ [0, 1], f \} \setminus \hat{B}$
Next we show that $\hat{A}$ is a metric attractor by showing that

$$\lambda \left[ B(\hat{A}) \right] = 1.$$ 

**Lemma A.8.** $\lambda \left[ B(\hat{A}) \right] = 1$

**Proof.** We show that $\lambda [\hat{B}] = 0$ then since $B(\hat{A}) = \lim \{ [0, 1], f \} \setminus \hat{B}$ the result will follow. First notice that for each $\hat{x} \in \hat{B}$, $x_1 \in B = \bigcup_{i=0}^{n-1} B_i$. So there is some specific interval, say $B_i$ containing $x_1$. This gives a decomposition of $\hat{B}$ into sets $\hat{B}_i$, $0 \leq i \leq n - 1$, by the first coordinate:

$$\hat{B} = \bigcup_{i=0}^{n-1} \hat{B}_i$$

where $\hat{B}_i = \{ \hat{x} \in \hat{B} : x_1 \in B_i \}$ for each $0 \leq i \leq n - 1$.

Consider first $\hat{B}_{n-1}$. Each $\hat{x} \in \hat{B}_{n-1}$ has the property that $x_1 \in B_{n-1}$, $x_2 \in B_{n-2}$, $x_3 \in B_{n-3}$, $\ldots$, $x_n \in B_0, x_{n+1} \in B_{n-1}$, $\ldots$. This implies that $\hat{x} \in \hat{B}_{n-1}$ if and only if $\hat{x}$ has backwards itinerary of the form

$$\ldots V s_3 V s_2 V s_1 V.$$ 

for some sequence, $(\ldots s_3, s_2, s_1)$, in 0’s and 1’s and word $V$ that encodes the path of the intervals $B_i$. So $\hat{B}_{n-1}$ has the property that all of its points have infinitely many restrictions on their backward itinerary. Hence $\lambda [\hat{B}_{n-1}] = 0$.

Next notice that $f^j(B_{n-j-1}) = B_{n-1}$, and so $\sigma^j(\hat{B}_{n-1}) = \hat{B}_{n-j-1}$ for all $0 \leq j \leq n - 1$. It follows that for all $0 \leq j \leq n - 1$, $\hat{B}_{n-j-1}$ has the property that all of its points have infinitely many restrictions on their backward itinerary. Thus $\lambda [\hat{B}_i] = 0$ for all $0 \leq i \leq n - 1$. Hence $\lambda [\hat{B}] = 0$ and $\lambda [B(\hat{A})] = 1$.

The theorem now follows since $\lambda [B(\hat{A})] = 1$, and from the discussion before the theorem we see that $\sigma|_{\hat{A}}$ is topologically transitive.
Proof of Lemma 7

By definition of Type B maps and attractors, $f$ is unimodal (piecewise monotonic), surjective and transitive on $[0, 1]$. Then, by Schultz’ Theorems 9 and 11, $f$ is topologically conjugate to a “restricted tent map” $T_s: [0, 1] \rightarrow [0, 1]$

$$T_s = \begin{cases} sx - s + 2 & \text{for } 0 \leq x < 1 - 1/s \\ s - sx & \text{for } 1 - 1/s \leq x \leq 1 \end{cases}$$

with slope $s$ greater or equal to $\sqrt{2}$. This means that there exists a homeomorphism $\vartheta: [0, 1] \rightarrow [0, 1]$ such that $\vartheta \circ f = T_s \circ \vartheta$. Next, by Schultz’ Theorem 9, a restricted tent map $T_s$ is topologically exact if, and only if, $s > \sqrt{2}$, which occurs if, and only if, $T_s(0) < p$ ($p = \vartheta(\bar{x})$ being the unique fixed point of $T_s$ and $\bar{x}$ the only fixed point of $f$). But $f(0) < \bar{x}$ implies $T_s(0) < p$ and therefore $T_s$ is topologically exact and, since topologically exactness is preserved by conjugacy, so is $f$.

Proof of Theorem 6.

In view of Lemma 7, it will be sufficient to prove the statement of the theorem for a unimodal map $f: [0, 1] \rightarrow [0, 1]$ with $f(1) = 0$ which is topologically exact on $[0, 1]$.

To begin we mention some well known properties of the one-sided shift space $\{0, 1\}^\mathbb{N}$ with its shift map $\sigma$ and the inverse limit space and its shift homeomorphism $\sigma^\mathbb{N}$. The interested reader should consult Walters (op. cit. §1.5), for background on ergodic theory and Ingram (2000a and 2000b) for background on inverse limit theory.

First, it is well-known that $\sigma$ is a measure-preserving transformation and it is ergodic on $\{0, 1\}^\mathbb{N}$. Every finite word, $w_0w_1 \ldots w_k$, in 0 and 1 corresponds to a basic open set, a so-called cylinder set, in $\{0, 1\}^\mathbb{N}$ defined by $\{(s_0s_1 \ldots) \in \{0, 1\}^\mathbb{N} : s_0 = w_0, s_1 = w_1, \ldots s_k = w_k\} = \{w_0\} \times \{w_1\} \ldots \{w_k\} \times \{0, 1\}^\mathbb{N}$, which we denote by $\{w_0w_1 \ldots w_k\} \times \{0, 1\}^\mathbb{N}$.

Notice that

$$\sigma^{-1}(\{w_0w_1 \ldots w_k\} \times \{0, 1\}^\mathbb{N}) = \{0, 1\} \times \{w_0w_1 \ldots w_k\} \times \{0, 1\}^\mathbb{N}.$$
In general

\[ \tilde{\sigma}^{-j}(\{w_0w_1 \ldots w_k\} \times \{0, 1\}^N) = \{0, 1\}^j \times \{w_0w_1 \ldots w_k\} \times \{0, 1\}^N. \]

Notice that the set

\[ \mathcal{A}_{w_0w_1 \ldots w_k} = \bigcup_{j=0}^{\infty} \tilde{\sigma}^{-j}(\{w_0w_1 \ldots w_k\} \times \{0, 1\}^N) \]

is \( \tilde{\sigma} \)-invariant, i.e.

\[ \tilde{\sigma}^{-1}(\mathcal{A}_{w_0w_1 \ldots w_k}) = \mathcal{A}_{w_0w_1 \ldots w_k}. \]

Since \( \hat{\lambda} \) is ergodic with respect to \( \tilde{\sigma} \), it must be the case that \( \hat{\lambda}(\mathcal{A}_{w_0w_1 \ldots w_k}) \) is either 0 or 1. Since \( \mathcal{A}_{w_0w_1 \ldots w_k} \) contains a non-empty open set it is the case that \( \mathcal{A}_{w_0w_1 \ldots w_k} \) has \( \hat{\lambda} \)-measure 1.

Now we consider the inverse limit space and its shift homeomorphism. By definition \( \sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \). So if \( (x_1, x_2, \ldots) \in \lim \{[0, 1], f\} \) then \( \sigma^{-1}(x_1, x_2, \ldots) = (f(x_1), x_1, x_2, \ldots) \). Let \( \hat{U} = \pi_n^{-1}(U) \). Then by properties of the inverse limit space \( \pi(U) = f^{n-1}(U) \) while \( \sigma^{-1}(\hat{U}) = \pi_{n+1}^{-1}(U) \) has \( \pi \)-image \( f^n(U) \). In general \( \sigma^{-j}(\hat{U}) = \pi_{n+j}^{-1}(U) \) and this set has \( \pi \)-image \( f^{n+j-1}(U) \). These facts will be used extensively in the proof of the next lemma.

**Lemma A.9.** Let \( f : [0, 1] \to [0, 1] \) be a unimodal, exact (l.e.o.) map with \( f(1) = 0 \). Let \( \hat{U} = \pi_n^{-1}(U) \) be a basic open set in \( \lim \{[0, 1], f\} \). Then

\[ \bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{U}) \]

has \( \lambda \)-measure 1.

**Proof.** By definition of \( \hat{U} = \pi_n^{-1}(U) \) we see that

\[ \pi(\hat{U}) = f^{n-1}(U) \]
and \( U \) is an open set in \([0, 1]\). Moreover, \( \sigma^{-1}(\hat{U}) = \pi_{n+1}^{-1}(U) \), and in general
\[
\sigma^{-j}(\hat{U}) = \pi_{n+j}^{-1}(U).
\]
Notice also that
\[
\pi(\sigma^{-j}(\hat{U})) = f^{n+j-1}(U).
\]
Since \( f \) is topologically exact, there is an integer \( J \) such that \( f^J(U) = [0, 1] \). Let \( M = \max\{J, n\} \). Then \( f^M(U) = [0, 1] \) and \( [x^*, 1] \subseteq f^{M-1}(U) \). Hence if we let \( j = M - n + 1 \) then
\[
\pi(\sigma^{-j}(\hat{U})) = f^{n+j-1}(U) = f^M(U) = [0, 1].
\]
There is a small subinterval \( V \) of \( U \) that has the property that \( f^{M-1}(V) = [x^*, 1] \) and \( f^{M-1}|_V \) is one-to-one. So
\[
\hat{V} = \pi_{M+1}^{-1}(V) \subseteq \pi_{M+1}^{-1}(U) = \sigma^{-j}(\hat{U})
\]
and
\[
\pi(\hat{V}) = f^M(V) = [0, 1]
\]
while
\[
\pi_2(\hat{V}) = f^{M-1}(V) = [x^*, 1].
\]
In fact since \( f^{M-1}|_V \) is one-to-one, there is a unique finite word, \( w_1 \ldots w_{M-1} \), such that \( f^{-1}_{w_1 \ldots w_{M-1}}([x^*, 1]) = V \), where by \( f^{-1}_{w_1 \ldots w_{M-1}} \) we mean the composition of the branches of the inverse of \( f \), \( f_L^{-1} \) and \( f_R^{-1} \) (coded as 0 and 1, respectively) in the pattern of the word \( w_1 \ldots w_{M-1} \). Since \( [x^*, 1] = f_R^{-1}([0, 1]) \), we see that
\[
f^{-1}_{1w_1 \ldots w_{M-1}}([0, 1]) = V.
\]
Notice also that \( \hat{V} \) satisfies the assumptions of Lemma A.2. Hence
\[
H^{-1}(\hat{V}) = f^M(V) \times \prod_{i=1}^M L_i \times \{0, 1\}^N
\]
where $\prod_{i=1}^{M} L_i = 1w_1 \ldots w_{M-1}$ and $f^M(V) = [0, 1]$. So

$$H^{-1}(\hat{V}) = [0, 1] \times \{1w_1w_2 \ldots w_{M-1}\} \times \{0, 1\}^N.$$ 

Now consider $\sigma^{-1}(\hat{V})$. Notice that

$$\pi(\sigma^{-1}(\hat{V})) = [0, 1] = \pi_2(\sigma^{-1}(\hat{V}))$$

while

$$\pi_3(\sigma^{-1}(\hat{V})) = [x^*, 1]$$

and

$$\pi_{M+2}(\sigma^{-1}(\hat{V})) = V.$$ 

This implies that

$$H^{-1}(\sigma^{-1}(\hat{V})) = [0, 1] \times \{0, 1\} \times \{1w_1w_2 \ldots w_{M-1}\} \times \{0, 1\}^N$$

and in general we have

$$H^{-1}(\sigma^{-j}(\hat{V})) = [0, 1] \times \{0, 1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0, 1\}^N.$$ 

Since

$$H^{-1} \left( \bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V}) \right) = \bigcup_{j=0}^{\infty} H^{-1}(\sigma^{-j}(\hat{V}))$$

and since

$$\bigcup_{j=0}^{\infty} [0, 1] \times \{0, 1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0, 1\}^N$$

$$= [0, 1] \times \left( \bigcup_{j=0}^{\infty} \{0, 1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0, 1\}^N \right),$$

we can compute $\lambda \left[ \bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V}) \right]$ by considering

$$\lambda \times \hat{\lambda} \left[ [0, 1] \times \left( \bigcup_{j=0}^{\infty} \{0, 1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0, 1\}^N \right) \right].$$
Clearly this is simply
\[ 1 \cdot \hat{\lambda} \left( \bigcup_{j=0}^{\infty} \{0,1\}^j \times \{w_1 \ldots w_{M-1}\} \times \{0,1\}^\mathbb{N} \right) \]

which is \( \hat{\lambda}(A_{1w_1 \ldots w_{M-1}}) \) in the notation of the discussion before the lemma. Again since \( \hat{\lambda} \) is ergodic and \( A_{1w_1 \ldots w_{M-1}} \) is \( \sigma \)-invariant and contains a non-empty open set, \( \hat{\lambda}(A_{1w_1 \ldots w_{M-1}}) = 1 \). Hence
\[ \frac{\lambda}{\hat{\lambda}} \left( \bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V}) \right) = 1 \]

and since \( \hat{V} \subseteq \hat{U} \) the lemma follows.

**Lemma A.10.** The set \( \hat{D} \) of points with a dense \( \sigma \)-orbit has
\[ \frac{\lambda}{\hat{\lambda}} (\hat{D}) = 1. \]

**Proof.** Let \( \{U_m\}_{m \in \mathbb{N}} \) be a countable collection of basic open sets of \( \lim \{[0,1], f\} \) that generates the topology on \( \lim \{[0,1], f\} \). Then it is easy to see that \( \hat{x} \) has a dense \( \sigma \)-orbit if, and only if,
\[ \hat{x} \in \bigcap_{m \in \mathbb{N}} \left( \bigcup_{j=0}^{\infty} \sigma^{-j}(U_m) \right). \]

By the previous lemma we have that each set \( \bigcup_{j=0}^{\infty} \sigma^{-j}(U_m) \) has \( \lambda \)-measure 1. Hence the intersection of all such sets has \( \lambda \)-measure 1. Therefore \( \hat{D} \) has \( \lambda \)-measure 1.

The proof of Theorem 6 can now be completed. Let \( K \sqsubset \lim \{[0,1], f\} \), and let \( B(K) \) be the basin of attraction for \( K \) under \( \sigma \). Notice that \( B(K) \sqsubset \lim \{[0,1], f\} \setminus \hat{D} \). Since \( \frac{\lambda}{\hat{\lambda}} (\hat{D}) = 1 \) we see that \( \frac{\lambda}{\hat{\lambda}} (B(K)) = 0 \). Thus \( K \) is not a metric attractor under \( \sigma \).
REFERENCES


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FOOTNOTES

1 The difference between the scope and method of the two approaches is best illustrated by the results of their applications to the same model – the leisure–consumption OLG model à la Grandmont (1985), which we label a “Type A” model. In our MR07 and here, we consider the set of all possible intertemporal perfect foresight equilibria and prove that, independently on the dynamical properties of the corresponding backward dynamics, the subset of equilibria converging forward in time to the “non–monetary” steady state is generic in a topological (MR07) as well as in a metric sense (this paper). On the contrary, GHTV consider the subset of equilibria that do not converge to the “non–monetary” equilibrium and show that, if the backward dynamics are chaotic and have a homoclinic orbit, the forward orbits generated by a conveniently restricted IFS converge to a fractal attractor.

2 For a more detailed discussion, see also MR07 quoted. Notice that, for our present purpose, the leisure–consumption OLG model à la Grandmont is perfectly equivalent to the model of pure exchange discussed, in various versions, by Samuelson (1958), Gale, op.cit., Benhabib and Day, op.cit. and many others. The former model can be transformed into the latter by replacing the hypothesis of variable labor supply and production with the alternative hypothesis of no–production and given endowments of the consumption good. The resulting dynamical equations would be formally identical in the two cases.

3 In particular, we, like Grandmont, assume that $V'(z) > 0, U'(\zeta) > 0; V'' < 0, U''(\zeta) < 0; \lim_{z \to -1} V'(z) = \infty; \lim_{\zeta \to -1} U'(\zeta) = \infty$.

4 In our notation the classical or the Samuelson case obtains if $V'(0) \geq U'(0)$, respectively.

5 The term “inverse” refers to the fact that $f$ maps each factor space $X_i, i \geq 2$ to its antecedent in the space sequence.

6 In simple words, the maps $\pi_i$ takes an element of the inverse limit space (an infinite forward–moving sequence) and returns its $i$th coordinate.

7 The inverse of the shift map, $\sigma^{-1}$ acts on a sequence $(x_1, x_2, \ldots)$ by replacing each term by its image under $f$. Thus, if $f$ is a BD map, the dynamics generated by
iterations of $\sigma^{-1}$ move backward in time. 

8 For example: suppose $A = \{x, y, z\}$. Then $\hat{A} = \{\hat{z}, \hat{y}, \hat{x}\}$, with

\[
\hat{z} = \{z, y, x, z, y, x, \ldots\}
\]

\[
\hat{y} = \{y, x, z, y, x, z \ldots\}
\]

\[
\hat{x} = \{x, z, y, x, z, y, \ldots\}
\]

Thus, $\sigma(\hat{z}) = \hat{y}; \sigma(\hat{y}) = \hat{x}; \sigma(\hat{x}) = \hat{z}$, and $\pi(\hat{A}) = A$. The interesting special case $n = 1$ arises when $A$ consists of a fixed point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$. In this case, $\hat{A}$ consists of a single sequence $\hat{x} = (\bar{x}, \bar{x}, \ldots)$.

9 We adopted this characterization of chaos, first suggested by Touhey (1997), because it suits our present purpose very well and immediately leads to the useful Corollary 1. On the other hand, Touhey also proved that his definition is equivalent (in the “iff” sense) to the more common Devaney’s definition. A map $f$ is said to be chaotic on a metric space $X$ in the sense of Devaney if it has the following properties:

1. $f$ is t.t. on $X$; 2. the periodic orbits of $f$ are dense in $X$; and 3. $f$ has sensitive dependence on initial conditions. It has been proved by Banks et al. (1992) that property 3 is redundant since it is implied by the other two. Moreover, Vellekoop and Berglund (1994) proved that, on intervals, property 1 alone implies the other two. It is also known that, on compact sets, Devaney–chaos implies Li and Yorke–chaos.

10 We are indebted to Artur Avila for discussing with us the properties of q.q. maps.

For a thorough analysis of this family of maps, see Avila, Lyubich and de Melo (2004). Notice that, in specific applications, proving that $Sf < 0$ may be the easiest way to establish that $f$ is q.q.. For example, if in the definition of the map $G$ of equation (7), we put $v(\zeta) = \tilde{r} + \zeta$ and $u(\zeta) = -r \zeta e^{-(\tilde{I} + \zeta)}$, with $r > 0$, we can show that $SG(\zeta) < 0$ for all choices of $\tilde{r}, \tilde{I}, r$. Also, it can be shown that if $v(\zeta) = (\tilde{I} + \zeta)^{1-\alpha_1}/(1 - \alpha_1)$ and $u(\zeta) = (\tilde{I} + \zeta)^{1-\alpha_2}/(1 - \alpha_2)$, $SG(\zeta) < 0$ if $\alpha_1 \leq 1$ and $\alpha_2 \geq 2$ (see, with different notation, Grandmont (1985, p. 1026)).

11 In the Samuelson case for which the young agent’s excess demand is at most zero, first period utility saturation is not relevant.

12 Notice that the inverse limit space of a Type B map can always be constructed from a Type A map $f$, simply taking the so–called “core map”, i.e. the restriction of $f$ to the
subinterval \([f^2(x^*), f(x^*)] \subset [0, 1]\). Thus, the space of forward admissible sequences generated by a Type A map always contains a subset \(R\) of “Type B sequences”, but, as we shall see, for Type A maps \(R\) has zero measure. On the other hand, for a Type B map, the set \(R\) include all the forward admissible sequences.


14 Points \(\zeta \in (1, \zeta_{\text{MAX}}]\) have a non-empty counter-image under \(F_\mu\) too, but they must be discarded since, for those values of \(\zeta\) the marginal utility is negative.

15 A recent, simpler proof of this fact can be found in Kraft (1999).

16 The probability spaces \((X_1, B_1, \mu_1)\) and \((X_2, B_2, \mu_2)\), are said to be isomorphic if there exist \(M_1 \in B_1\) and \(M_2 \in B_2\), with \(\mu_1(M_1) = 1 = \mu_2(M_2)\) and an invertible, measure-preserving map \(\phi: M_1 \to M_2\) (where the space \(M_i\) is assumed to be equipped with the \(\sigma\)-algebra \(M_i \cap B_i = \{M_i \cap B | B \in B_i\}\)).

17 The PIR – renamed “Principle of Indifference” by John Maynard Keynes in his Treatise on Probability (1921) – is a rule for assigning probabilities under ignorance and can be seen as a special case of the Bayesian “prior distribution” in the absence of background information.

18 The \(\lambda\) measure is not the only measure possessing these properties, but what makes it uniquely appealing is that \(\lambda\) is a natural application of certain basic principles of probability theory (Principle of Insufficient Reason/Maximum Entropy) to the situation described by the economic models discussed in the paper.

19 We omit the (trivial) case in which the unique “monetary” stationary state \(\bar{x}\) is located on the left of, or on the critical point. In this case, \(\hat{x} = (\bar{x}, \bar{x}, \ldots)\) is the only existing forward admissible orbit.

20 See Ingram (1995), in particular Theorems 2 and 3. Moreover, remember that, for any \(n > 1\), \(\lim \{X, f\}\) is homeomorphic to \(\lim \{X, f^n\}\).

21 See Avila, Lyubich and de Melo (2004); Avila and Moreira (2005a, 2005b).

22 In what follows, we will use the term “chaotic” instead of the less common “stochastic”. Notice that an attractor which is stochastic in the sense of Avila & Co., is also chaotic in the sense of Devaney/Touhey.

23 To see this consider the proof contained in Devaney (2003, §1.7) of the fact that the itinerary mapping is a homeomorphism from his set \(\Lambda\) onto \(\{0, 1\}^\mathbb{N}\). The only
assumptions he uses to prove that the map is continuous and 1-1 are that \( \Lambda \) does not contain the critical point or its preimages and that it has slope larger than 1. Since our set \( \Lambda \) also has these properties we see that \( i \) is continuous and 1-1. It follows that \( i \) is a homeomorphism onto the image of \( \Lambda, i(\Lambda) \).

A continuous map \( f: [0, 1] \to [0, 1] \) is said to be topologically exact, or locally eventually onto (l.e.o.) if for any open, nonempty set \( V \subset [0, 1] \) there exist \( n \geq 0 \) such that \( f^n(V) = [0, 1] \). Topological exactness can be looked at as a strong form of indecomposability that implies (but it is not implied by) transitivity.

Since Type C maps has been already covered completely in MR07, here we only deal with Type A and B maps.

We denote by \( \sigma \) the one-sided shift map on the space of sequences of 2 symbols \( \{0, 1\} \), sometimes called (one-sided) “Bernoulli shift”. The action of the map consists in deleting the first element and shifting the remaining sequence one step to the left. Notice that \( \sigma \) is a two-to-one map and therefore not invertible, whereas, as we have already explained, \( \sigma \) is a homeomorphism.