Discrete and Continuous Time in Optimal Growth Models

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Continuous or Discrete Time?

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Continuous or Discrete Time?

- Most dynamical models in economics are represented by systems of difference or differential equations.
- Sometimes the choice between one or the other characterization of time is suggested by the nature of the problem at hand. More often, however, it depends on the author’s undisclosed preferences or the mathematical tools s/he possesses.
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This is surprising since continuous– and discrete–time OG models depicting altogether similar economies may generate entirely different dynamics.
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This is surprising since continuous– and discrete–time OG models depicting altogether similar economies may generate entirely different dynamics.

The simpler and more striking example is provided by one–dimensional (1D) OG models on which we now focus.
Continuous–time

\[
\begin{align*}
\max & \quad \int_0^\infty e^{-rt} V[x(t), \dot{x}(t)] dt \\
(\dot{x}, x) & \in T \subset X \times Y, x(0) = x_0
\end{align*}
\] (1)

where \( V(x, \dot{x}) \) is the concave objective function, \( X \) and \( Y \) are subsets of \( \mathbb{R} \) and \( r \) is the discount rate
Continuous–time

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Under generally assumed conditions, from (1) we can derive a unique differential equation, the policy function

\[
\dot{x} = f(x), \quad x(0) = x_0
\] (2)
Discrete–time

\[
\max \sum_{n=0}^{\infty} \beta^n V(x_n, x_{n+1}) \\
(x_n, x_{n+1}) \in T' \subset X \times X
\]  

where \( \beta = 1/(1 + r) \) is the discount factor
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Analogously, from (3) we can derive the policy function:

$$x_{t+1} = G(x_t)$$ (4)
Optimal Growth, II

Discrete–time

\[
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(4)

- Elementary mathematical considerations indicate that, if \( X \subset \mathbb{R} \), system (2) must necessarily have trivial dynamics. On the contrary, even in the 1D case, iterations of the map \( G \) in (4) may generate complex or chaotic dynamics.
From (3) we can derive the following one–parameter family of models, differing only for the period of time, $\Delta$, that separates two successive states of the system,

$$\max_z \sum_{n=0}^{\infty} \beta(\Delta)^n \Delta V_{\Delta}(x, z) \Delta$$  \hspace{1cm} (5)

with policy function

$$x_{n+1,\Delta} = h_{\Delta}(x_{n,\Delta})$$  \hspace{1cm} (6)

where

- $t = n\Delta, \ n \in \mathbb{Z}$
- $V_{\Delta}(x, z) = V\left(x_{n,\Delta}, \frac{x_{n+1,\Delta} - x_{n,\Delta}}{\Delta}\right)$
- $x_{n,\Delta} = x(n\Delta)$
- $\lim_{\Delta \to 0} z = \dot{x}(t)$
- $\beta(\Delta) = (1 + r\Delta)^{-1/\Delta}$
- $\lim_{\Delta \to 0} \beta(\Delta) = e^{-r}$
We take as a benchmark the 1D, discrete–time OG model discussed by Denekere and Pelikan (DP) (1986), which is perhaps the first fully investigated example of chaos for that class of models.
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The objective function $V_\Delta(x, z)$ is chosen so that the member of the $\Delta$–family for which $\Delta = 1$ (i.e., the length of the period is equal to the unit of measure of time) corresponds to the DP benchmark model, whereas in the limit for $\Delta \to 0$, equations (5) and (6) collapse to the continuous time forms (1) and (2).
Notice that the unit or measure of time is arbitrary. When we change it, nothing change in the structure of the models, except that all the quantities with a time dimension change their numerical values accordingly.
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If we want to discuss the effect of structural changes (i.e., of changes that do not depend on modifications of the unit of time) of one of those quantities, such as $\Delta$, $V(x, y)$ or $r$, the best strategy is to fix the unit of measure of time once and for all (say, one year) and let only $\Delta$ vary.
In the economic literature on growth, the occurrence of chaos is linked to the parameter $r$, the rate of discount (or, equivalently, the discount factor $1/(1 + r)$), with $\Delta = 1$ fixed.
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We discuss the effect on the dynamic behavior of (6) of changes of $\Delta$, from 1 to 0, first with fixed $r$ and then when both $r$ and $\Delta$ are allowed to change.
In order for this exercise to make good economic sense, the objective function $V_\Delta(x, z)$ must retain all the required properties, in particular concavity, for $\Delta \in (0, 1]$.
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Applying Bellman Dynamic Programming, the problem can be written as follows:

$$W_\Delta(x) = \max_z \{ V_\Delta(x, z)\Delta + (1 + r\Delta)^{-1}(W_\Delta(\Delta z + x)) \} \quad (7)$$
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The solution of (7) is given by a value function $W_{\Delta}(x)$ and a function $z = h_{\Delta}(x)$ such that

$$W_{\Delta}(x) = V_{\Delta}[x, h_{\Delta}(x)] + (1 + r\Delta)^{-1}W_{\Delta}[x + \Delta h_{\Delta}(x)] \quad (8)$$
From (8) and the definition of \( z \), and simplifying notation (\( x_n \) instead of \( x_{n,\Delta} \)) the policy function of (8) can be written as:

\[
x_{n+1} = G_\Delta(x_n) = A_\Delta x_n - B_\Delta x_n^2
\]  

(9)

where \( G_\Delta(x) = \Delta h_\Delta(x) + x \)

Putting \( y = B/Ax \), for all \( \Delta > 0 \) system (9) can be replaced by a simpler, dynamically equivalent system defined by the standard logistic equation

\[
y_{n+1} = \mu_\Delta y_n(1 - y_n)
\]  

(10)

with \( \mu_\Delta = A_\Delta \)
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DP showed that, for a more or less plausible set of parameters, $A_1 = B_1 = 4$ for which the dynamics of (9)–(10) is known to be chaotic.
A $\Delta$–bifurcation exercise

- Fix the parameters (in particular $r$) so that, for $\Delta = 1$, we have $\mu = 4$ and chaos.
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- As $\Delta$ declines from 1 to an arbitrarily low value $> 0$, $\mu$ monotonically declines from 4 to 1 and we have a classical period-halving bifurcation (see Figure 1). When $\mu$ becomes smaller than 3, the typical orbit converges to a stable fixed point.
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- As $\Delta$ declines from 1 to an arbitrarily low value $> 0$, $\mu$ monotonically declines from 4 to 1 and we have a classical period-halving bifurcation (see Figure 1). When $\mu$ becomes smaller than 3, the typical orbit converges to a stable fixed point.
- Figure 2, shows a two–parameter bifurcation scenario, the two parameters being $\Delta$ and the interest rate $r$. Different colors correspond to cycles of different periods. The system dynamics becomes more complex as we move in the “north–east” direction toward the black area.
Figure 1. Period–halving bifurcation
Figure 2. Two–parameter ($\Delta - r$) bifurcation
Continuous–time

- As $\Delta \to 0$, system (3) $\to$ system (1), with 
  \[ W_0(x) = \lim_{\Delta \to 0} W_\Delta(x), \quad V_0(x, \dot{x}) = \lim_{\Delta \to 0} V_\Delta(x, z) \]

- Write the present value Hamiltonian function

\[
H(x, q) = \max_{\dot{x}} \left\{ V_0(x, \dot{x}) + q \dot{x} \right\}
\]

(11)

where $q$ denotes the co–state variable

- Applying the Pontryagin Maximum Principle and the known fact that, along the optimal orbit it must be $q = W_0'(x)$, we obtain the policy function for problem (1)

\[
\dot{x} = f(x) = H_q(x, W_0'(x))
\]

(12)

which, in our case, happens to be equal to

\[
G_0(x) = \lim_{\Delta \to 0} h_\Delta(x)
\]

- System (12) has two equilibria for which $\dot{x} = 0$, $x_1 = 0$ and $x_2 > 0$, the former unstable and the latter stable

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Some conclusions and a dilemma

Simple calculations show that, in our model, for any arbitrarily large value of $r$, there exists a sufficiently low value of $\Delta$ such that the dynamics of (9)–(10) is simple. True, for any given value of $\Delta \in (0, 1]$, there exists a sufficiently large value of $r$ for which (9)–(10) generates chaos. However, the levels of $r$ needed to yield chaos are extremely large even for values of $\Delta$ relatively large (say, $0.5$) and they grow exponentially fast as $\Delta$ declines.

Thus, we have the following dilemma: to make large discount rates plausible, we should choose a large unit of time. However, this makes it unreasonable to choose $\Delta$ sufficiently close to one to make chaos possible.
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REFERENCES