Economics sometimes works in strange and nonlinear ways. Ronald Coase’s seminal article (1937) gave its author one-half of his Nobel Prize, although it was not considered an important contribution until the late 1960s. Léon Walras, when he wrote his *Éléments d’économie politique pure*, was not at all a prophet in his own country. It is possible to multiply such examples.

In this article, we want to show that an author’s true value may not be appreciated at the time of writing, even when that author is making an important contribution to demand theory.

In 1946 Jean Ville published (in *Les annales de l’Université de Lyon*) an article titled “Sur les conditions d’existence d’une ophélimité totale et d’un indice de prix.” This short article constitutes one of the more original French contributions to economics in the first part of the twentieth century, and it is strange that in France economists have forgotten it. In 1938 Paul Samuelson published three articles on the theory of utility that initiated the revealed preference theory. In these contributions,
Samuelson, using the “weak axiom of revealed preferences” (WARP), established a theoretical result that restated some major results of the ordinal utility theory without using any nonobservational concepts. He used only the notions of prices, quantities, and income. In a similar way, Ville ([1946] 1951–52, 123) proposes that we “study price-formation directly, starting from hypotheses which, fundamentally, entail the existence of $\Phi$, but which are more directly expressible as functions of data obtainable from observation.” In fact, in the four assumptions he introduces, he uses prices, quantities, and income exactly in the same way Samuelson does in his revealed preference theory. However, while Samuelson, in his first articles (1938a, 1938b, 1938c), does not consider integrability as a relevant issue, Ville demonstrates that the inexistence of a closed line, along which the sum of the variations of the quantities multiplied by their prices keeps the same sign, is a necessary and sufficient condition for this sum to have an integrating factor. Samuelson revised his views in 1950, and Hendrik Houthakker (1950) evaluated Samuelson’s theory on the basis of the integrability problem.

Houthakker’s proof is based on a discrete analysis that later transforms into a continuous one. He is considered as theorizing the “strong axiom of revealed preferences” (SARP). Ville directly solves the integrability problem using continuous analysis. Indeed, the shared idea is generally that the “Ville axiom of revealed preferences” (VARP) can be transformed into the SARP by means of a small change. Leonid Hurwicz and Marcel K. Richter (1979) show, however, that these two axioms are not linked to the same properties of the Antonelli matrix. SARP implies symmetry as well as the semidefiniteness of the Antonelli matrix, whereas the VARP lacks this last property. One of our aims is to explain this phenomenon and to analyze its consequences.

The remainder of this article is organized as follows. Section 1 compares Samuelson’s 1938 articles and Ville [1946] 1951–52, emphasizing their common aim to build a theory of consumption based only on observable variables. In section 2 we present Houthakker’s discussion of Samuel-

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3. “Nous nous proposons dans le présent article d’étudier la formation des prix directement, à partir d’hypothèses revenant au fond à supposer l’existence de $\Phi$ [fonction d’ophélimité totale], mais plus directement exprimables en fonction des données fournies par l’observation” (Ville 1946, 32).

4. This shared interpretation is, however, challenged. Philippe Mongin (2000) defends the idea that Samuelson’s contributions, even in 1938, do not consider the integrability problem an unimportant one. Moreover, according to Mongin, Samuelson was not, in his 1938 papers, condemning the use of utility functions in economics.
son’s revealed preference theory, as well as Houthakker’s introduction of the SARP. In section 3 we discuss Hurwitz and Richter’s arguments. In section 4 we give some explanations for why Ville’s demonstration remained unknown until the 1970s. Section 5 concludes.

1. The Samuelson and Ville Analyses

Samuelson (1950, 358 n. 8) recognizes that his knowledge of Ville’s 1946 article is indirect and fuzzy: “There is a recent paper by (de) Ville which I know only from a brief review by K. Arrow in Mathematical Reviews, 1947.” Ville does not quote any economist (except François Divisia, on page 38) and not Samuelson’s work at all. An explanation of this lacuna can be found in the English translation of Ville’s article. Peter Newman writes:

M. Ville, a mathematician, first became interested in economics under rather peculiar circumstances. During his tenure of a professorship at Lyons, the shortage of academic staff forced him to lecture on “financial mathematics,” as well as on his regular subject—mechanics. He, therefore, read several works on economics, being much influenced by Pareto’s Cours d’Economie Politique. The theory of index numbers formed part of this course, and it was his attempt to put this difficult subject into some sort of order that led to this article. Primarily, he sought definite answers to the question: “When prices and incomes change, what can be said about the change in the individual’s standard of living?” (Ville [1946] 1951–52, 128n)5

We now compare Samuelson 1938a, 1938b, 1938c and Ville [1946] 1951–52 on the basis of their objectives, assumptions, and results and then introduce Houthakker’s analysis.

Partially Common Objectives

In 1938 Samuelson considered that the consumption theory based on the notion of utility progressively abandoned a whole part of its restrictive conditions: “(a) the assumption of linearity of marginal utilities; (b) the assumption of independence of utilities; (c) the assumption of the measurability of utility in a cardinal sense; and (d) even the assumption of an

5. In fact, Ville was aware of economics. Pierre Crépel told us that in Lyon economists and mathematicians knew each other well and often discussed economic problems.
integrable field of preference elements” (1938b, 61). It is clear that (d) implies that Samuelson can do without the integrability assumption. For him, the notion of utility is a psychological one, and its definition is empty and circular. “I propose, therefore that we start anew in direct attack upon the problem, dropping off the last vestiges of the utility analysis” (62). The “last vestiges” referred to John Hicks and R. G. D. Allen’s (1934a, 1934b) attempt to reconsider the value theory from the notion of the marginal rate of substitution. Samuelson (1938a, 345) writes in a seemingly neopositivistic flavor: “It is the purpose here to demonstrate that the utility analysis in its ordinary form does not contain empirically meaningful implications by which it could be refuted.”

In 1946 Ville analyzed price formation as well as the conditions of existence of an index for the price level (something lacking in Vilfredo Pareto’s *Cours d’économie politique* [1896] and his *Manuel* [1906]) and made the assumption, on the basis of observable data, that there exists a utility function. In fact, he thinks that the analysis of price formation in economics relies on the assumption that marginal utility functions define a relationship between the variation for satisfaction and the variation for consumed quantities. However, it is impossible to define a total utility function from those marginal utility functions, because any monotonic function with its partial derivatives proportional to the marginal utility functions has the same properties relative to this function. It is the reason why Ville assumes the existence of a total utility function and tries to define both its existence conditions and its properties.

Different Assumptions

In 1938 Samuelson’s arguments start from the three following assumptions:

I. Demand functions are given and individuals spend all their budgets.
II. These functions are homogenous of degree 0 in terms of the prices and income (this assumption corresponds to the second of Ville who considers that there is no money illusion).
III. If an individual selects batch one over batch two, he does not select two over one. (1938b, 65)

Samuelson (1938c) shows that, under some conditions, the third postulate implies the first two. This is the reason why the literature retains the

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6. This result has been known since Fisher 1892 and Pareto 1896.
“Samuelson postulate.” Not only is Samuelson reluctant to introduce the idea that the demand functions need to be integrable, but he considers that this condition is both unnecessary and unimportant. “Concerning the question of integrability I have little to say. I cannot see that it is really an important problem, particularly if we are willing to dispense with the utility concept and its vestigial remnants. . . . I should strongly deny, however, that for a rational and consistent individual, integrability is implied, except possibly as a matter of circular definition” (Samuelson 1938b, 68).

Ville ([1946] 1951–52, 123) introduces four “hypotheses relating to the behaviour of a buyer of directly consumable goods,”7 which are almost similar to Samuelson’s assumptions:

1. Quantities bought are only a function of the set of the prices and the income of the buyer.
2. The buyer is not a victim, prey to money illusion. His behavior is sensitive only to relative prices and his income (Ville writes in a note that this assumption is not necessary and shows that it is implied by the fourth one).
3. The third defines the consequences of a variation of the prices and income on the agent’s standard of living: if a buyer does not buy the same quantities after a change in prices and in his income, although he can do it, then his standard of living has risen.
4. If the prices and income have a cyclical pattern then the standard of living of a given individual cannot have increased over the cycle.

The last assumption is crucial because it permits demonstration of the fact that the sum of the price variations multiplied by the corresponding quantities has an integrating factor. The first two assumptions are common to Samuelson and Ville: the first, the idea that the consumed quantities are perfectly defined by prices and the income, and the second, the assumption that the demand functions are homogeneous of degree zero, in terms of the prices and income, are now classical ones. The third assumption contains the principle of revealed preferences and, at the same time, corresponds to the WARP of Samuelson 1950 because the modifications of the choices are linked to an increase of the “standard of living” (i.e., the satisfaction of the individual, supposed to be independent of prices). It is then impossible to take the opposite position, which is to prefer the second batch to the first in another situation, because the

7. “Hypothèses relatives au comportement d’un acheteur de biens directement consom- mables” (Ville 1946, 32).
standard of living has decreased. The fourth assumption contains a share of the SARP.

Indeed, the WARP stipulates that “if at the price and income situation A you could obtain the goods actually bought at a different point B and if you actually chose not to, then A is defined to be ‘revealed better than’ B. The basic postulate is that B is never revealed to be also ‘better than A’”; under SARP, “if A reveals itself to be ‘better than’ B, and if B reveals itself to be ‘better than’ C, and if C reveals to be ‘better than’ D, etc. . . . , then I extend the definition of ‘revealed preference’ and say that A can be defined to be ‘revealed to be better than’ Z, the last in the chain. In such a case it is postulated that Z must never also be revealed to be better than A” (Samuelson 1950, 370–71).

The problem lies in the fact that this second definition does not correspond to what Samuelson writes in 1938, because of the existence of Houthakker’s article.

Samuelson’s and Ville’s Results

The results of Samuelson 1938a, 1938b, 1938c are explicitly rooted in the era’s debates about consumer theory. The first aim of Samuelson’s theory, which is also Ville’s, is to show that it is possible to develop a consumer theory on the basis of the prices and income only. The second is that WARP in Samuelson 1938c is sufficient even if he writes that it is not possible to rediscover at all Eugen Slutsky’s results.8

Samuelson introduces a set of goods \((q_1, \ldots, q_n)\) with a set of corresponding prices \((p_1, \ldots, p_n)\), then another \((q'_1, \ldots, q'_n)\) of goods and a set of corresponding prices \((p'_1, \ldots, p'_n)\). Let \(p \cdot q = p_1 q_1 + \ldots + p_n q_n\) and \(p' \cdot q' = p'_1 q'_1 + \ldots + p'_n q'_n\). If the cost \(p \cdot q\) is lower or equal to \(p' \cdot q'\), that means that the individual could have bought the quantities of the second set of goods at the prices corresponding to the first set and that he did not do so. This is expressed by

\[
p \cdot q \geq p' \cdot q'.
\]  

In other words, the first batch has been revealed to be preferred to the second. On this basis Samuelson introduces the antisymmetry property of this relationship showing that

\[
p' \cdot q' \geq p' \cdot q.
\]

8. “For example one arrives at most of the properties of the Slutsky’s matrix” (Samuelson 1950, 370 n. 1).
implying that the second batch is preferred to the first, which contradicts $p \cdot q \geq p' \cdot q'$. Then the reverse inequality must hold:

$$p' \cdot q' < p' \cdot q.$$  \hspace{1cm} (2)

From (1) and (2), Samuelson derives his postulate 3. As written above, Samuelson (1938c) shows that this postulate implies the two first assumptions. Thanks to this postulate, he recovers the Georgescu-Roegen 1936 results, and he completely specifies the restrictive conditions on the demand functions.

Ville's results are in fact of two types. He first demonstrates the conditions of existence of a total utility function (*fonction d'ophélimité totale*) defined in terms of prices and income, on the basis of assumptions 1–4. Thus he again finds, but in a different way, Samuelson's (1938b) principle of revealed preference for one set of consumed goods compared with a second one. He takes two arbitrarily close situations in terms of prices, quantities, and income and shows that the cost in the second situation of the quantities consumed in the first one depends on the sign of the sum of the prices, multiplied by the variations of the corresponding quantities. If this sum is positive (respectively negative), the individual's standard of living has increased (respectively decreased). This reasoning is exactly the same as Samuelson's, but Samuelson finds that this sum is zero and then bases his arguments on the asymmetrical property of the relationship between the batches of goods. Ville then demonstrates (see appendix A) that his assumption 4 implies integrability, and by introducing the integrating factor of the sum of the prices multiplied by the variations of the corresponding quantities, he shows (in only one page) that there exists a total utility function that respects the "classical" results in terms of utility maximization. One interesting consequence of this proof is the introduction of symmetric expressions in terms of prices and quantities. He also shows that, depending on the existence of the total utility function, its lack of variation over time and the correct value for the integrating factor, it is then possible to calculate an index of the price level and of the standard of living.

2. Houthakker and the Integrability Question

Samuelson's results opened a new research program (Wong 1978), which enlarged considerably the limited viewpoint of his first articles. Considering

9. We will see below the consequences of this result.
that Samuelson’s weak axiom is not sufficient in the general case, Houthakker introduced, in 1950, the strong axiom of revealed preference to provide an axiomatic foundation for the existence of a utility function.

The main object of our investigation is to find a proposition which, apart from continuity assumptions, summarises the entire theory of the standard case of consumer’s behaviour. . . . Samuelson’s hypothesis does not satisfy this criterion, being only a necessary condition and not a sufficient one, for although it can be derived from utility considerations it does not entail integrability, which is an essential property of utility functions. (Houthakker 1950, 161)

Houthakker uses the first two axioms proposed by Samuelson 1938b and Ville [1946] 1951–52, but first introduces his strong axiom, so that these axioms seem to be embedded in his strong axiom, which seems to be equivalent to Ville’s third and fourth hypotheses:

If \( X_0, X_1, X_2 \ldots X_T \) is a sequence of batches of goods such as each batch is bought at prices \( P_0, P_1, P_2 \ldots P_T \) respectively, and if at least two of these batches are different, and if the cost of each batch \( X_t \) at price \( P_{t-1} \) is not greater than the cost \( P_{t-1} X_{t-1} \) of the preceding batch in the sequence \( X_{t-1} \) at the same prices, then the cost \( P_T X_T \) of the last batch \( X_T \) at prices \( P_T \) is less than the cost \( P_T X_0 \) of the first batch \( X_0 \) at the same prices.” (Houthakker 1950, 163)

In 1950 Samuelson recognized the importance of supposing the impossibility of closed paths: “I soon realised that this could carry us almost all the way along the path of providing new foundations for utility theory. But not quite all the way. The problem of integrability, it soon became obvious, could not yield to this weak axiom alone. . . . But no proof was forthcoming for all these years, until Mr. Houthakker’s paper arrived in the daily mail” (371). It is interesting to note here that W. J. Corlett and Newman (1952–53) do not refer to Ville’s article, although Newman did translate it in 1951–52. In fact, in criticizing Houthakker’s proof, they use Nicholas Georgescu-Roegen’s results (1936) to stress the necessity of transitivity to discuss the existence of indifference curves and to imply integrability.10

10. In fact, Georgescu-Roegen did not really show that transitivity implies integrability, but just that integrability is not sufficient to imply the existence of indifference curves: “The preceding analysis shows clearly that without the transitivity postulate the integral varieties and the indifference varieties are two distinct things. If a point of saturation exists, the indifference
Contrary to Samuelson’s reasoning, Ville’s proof does not refer to preferences or indifference, so it is not concerned by J. M. D. Little’s (1950) “semantic” critique. Ville’s proof stands completely apart from the debates between economists (he does not, for instance, suppose that the sum of prices weighted by changes in quantities is zero, which is implied by Samuelson’s third axiom, allowing for the definition of isoquants). Contrary to Samuelson, Ville does not have to compare his results with Georgescu-Roegen’s theory, and his proof is much more elegant from a mathematical point of view, as it contains implicitly the theory of ordinal utility without referring explicitly to it.\(^{11}\)

Despite these similarities, Ville’s axiom is not equivalent to Houthakker’s strong one: for differentiable demand functions, it is a consequence of the strong axiom. However, the strong axiom implies more than the integrability of demand functions: it also implies the semidefiniteness of the Antonelli matrix, which is equivalent to the weak axiom and thus is not sufficient to obtain Ville’s conditions. As shown by Leonid Hurwicz and Marcel K. Richter (1979), the a-cyclicity contained in Ville’s conditions is a necessary and sufficient condition for the symmetry of the Antonelli matrix, though it is not sufficient for its semidefiniteness and thus it does not imply the weak axiom, as will be discussed in the next section.

3. VARP, SARP, and WARP Reconsidered and the Role of Hurwicz and Richter

In their 1979 article, Hurwicz and Richter give a slightly more general version of Ville’s proof qualified as “a most important contribution to

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\(^{11}\) Hurwicz and Richter (1979, 607–8) note that this may lead to some misunderstanding of his results: “Because he uses preference terminology rather than revealed preference terminology, he might at first glance appear to employ circular reasoning, assuming a preference exists in order to prove a preference exists.”
the axiomatic of the consumer theory” (603). Their theorem 1 (p. 609) shows that if no continuous cycle exists in the neighborhood of some point, then the Antonelli matrix is symmetric in another neighborhood of the same point, thus implying the integrability of the corresponding demand functions. They show also the opposite: the symmetry of the Antonelli matrix around some point implies the absence of continuous cycles in the neighborhood of this point. The crucial consequence of their analysis is that Houthakker’s strong axiom, because it encompasses the weak axiom, also implies the semidefiniteness of the Antonelli matrix: the strong axiom is too strong because it is not equivalent to the sole symmetry of the Antonelli matrix, which is sufficient for obtaining integrability.

Ville’s axiom corresponds to the strong axiom in continuous terms: in SARP, every path composed of bundles \( q^1, \ldots, q^k \) and the vector of prices and income \((p^k, y^k)\) cannot contain any cycle of revealed preferences. This means that the following inequalities between consecutive situations

\[
P(\tau_2)[x(\tau_2) - x(\tau_1)] > 0 \quad (1a)
\]

cannot be verified all throughout the cycle, whatever it is. The continuous version of this assumption says that, throughout a continuous cycle, the inequality

\[
P(\tau)\cdot dx/dt(\tau) > 0 \quad (1b)
\]
is not always verified.

The simple fact of considering the continuous version of axiom (1b) makes the semidefiniteness of the Antonelli matrix vanish. Ville’s axiom thus appears as the minimal condition to imply integrability, while Houthakker’s proof is not satisfactory, since it assumes a stronger hypothesis. Therein lies the true difference between the two theories, not, as often mentioned, because inequalities are written in their strict or wide version.

One may note that the discrete version (1a) would be obtained by integrating the differential condition (1b) over finite intervals. In contrast, what proves Hurwicz-Richter equivalences is that this integration is possible only under some regularity condition of the corresponding demand functions, a condition given by the semidefiniteness of the Antonelli matrix. Therefore the discrete assumptions cannot be deduced from the continuous assumption, which is the only necessary assumption for integrability.

12. See also Hurwicz 1971 and Hurwicz and Uzawa 1971.
The first difficulty lies in the algebraic, or differential, form of the axioms. For the weak axiom, which compares only two situations, no difference exists between these two versions, but the difference is important for the strong axioms: Ville’s axiom is explicitly differential, relating to continuous preference cycles, while Houthakker compares a finite number of situations pushed to the limit by rather intricate reasoning. Also, the role of the continuity hypothesis is not clear in Houthakker’s analysis, as is also the case for the equivalences between the axioms and the properties of demand functions and the Slutsky or Antonelli matrices.

Second, the weak axiom (which is just equivalent to the positive semidefiniteness of the Slutsky matrix; see, e.g., John 1995) corresponds to a hypothesis of consistency (i.e., antisymmetry) between two situations \(x\) and \(x + dx\), that is, to the stability of preferences from a dynamic point of view. Ville’s axiom implies the symmetry of the Antonelli matrix, which makes it possible to integrate demand functions into a utility function, and thus the possibility of comparing all situations (for the same reason, Arrow [1959, 1974] has to add an assumption of comparability to his weak axiom to prove the existence of a utility; see also Campbell 1994).

Finally, SARP compares a much greater number of situations than WARP, thus justifying its name. Each of these two axioms corresponds to a different property of the Slutsky and Antonelli matrices, the empirical content of which is still largely unknown. A rapid combinatory analysis of the situations compared by these axioms tends to show that the strong axiom compares a number of situations in proportion \((n - 2)\), with \(n\) the number of bundles, to the number of situations compared by the weak axiom. However, an empirical analysis of rationality conditions in a Polish panel (Diaye, Gardes, and Starzec 2001) tends to show that the violations of WARP are as numerous as those of SARP and GARP (the generalized axiom defined by Sidney N. Afriat), which seems to indicate that their empirical contents are not ordered as would normally be assumed.

4. Why Was Ville’s Axiom Not Considered Earlier?

Why has there been this silence about Ville’s contribution, despite Arrow’s discussion of it as early as 1947 and its early translation by Newman? Its mathematical content seems easily understandable, even by economists, and so pure that, once understood, it is completely mastered. Moreover, it is not easy to find any internal logical contradiction, while the conver-
gence arguments of Houthakker’s article are highly contestable and have been contested (Corlett and Newman 1952–53; Stigum 1973), since Houthakker uses some continuity assumptions in his original proof that are not sufficiently defined (see especially pages 165–67 and the implicit assumption of uniform continuity, as in Stigum 1973, 411).

So, again, why wasn’t Ville’s contribution considered earlier? It does not seem necessary to introduce sociological explanations to answer that question. Rather, following Lakatos 1983, we introduce elements of external history, an external history that is necessary to complement the internal history of economics. And we see from the external history that there are two elements that account for the neglect of Ville’s achievement: Ville’s personality, and the French economic community. Those two tell us more about the attention (or lack thereof) paid to Ville than does the functioning of the international community of economists at the time.

With those two elements, we can explain the fact that Ville was not as well known in the economics community as Gerard Debreu (who also was trained at the Ecole Normale Supérieure), as well as the fact that his article was not really considered before the 1970s. But there is a third reason as well, and that is the relative novelty of the mathematical tools he used.

Some of Jean Ville’s Personal Characteristics

Ville was born in Marseille in 1910. He entered the Ecole Normale Supérieure (ENS) in 1929 as first in the competition, and was second in passing the French aggregation exam in 1932, which he considered a failure. After his military service he entered the Institut Français de Berlin and then obtained a fellowship at the University of Vienna, where he published his first papers on probability theory. At that time, he was a member of Karl Menger’s circle.

He was then hired by France’s National Center for Scientific Research (CNRS) in 1935 as a research associate, until 1938. In that year he obtained

13. Stigum (1973, 411) discusses the hypothesis that “the two sequences of income functions corresponding to Houthakker’s upper and lower sequences of offer curves possess subsequences which converge to a function that satisfies the ‘right’ differential equation.”

14. The Ecole Normale Supérieure, along with the Ecole Polytechnique, is France’s leading higher educational establishment. The aggregation is a national competitive exam that provides immediate access to a teaching position in the educational system.
his PhD in mathematics as well as a diploma in law. The title of his dissertation was “Etude critique de la notion de collectif” (“A Critical Study of the Notion of the Collective”), which also concerns probability theory.\footnote{15} In the early 1940s he obtained a position at the University of Lyon, where he taught physics, mathematics, and economics (mathematical finance). Using Pareto’s Manuel, he discovered that an important theorem was missing, which was necessary to give theoretical foundations to the existence of a price index. He then published his paper in 1946, which studies the consequences of the changes in prices and income on individuals’ standards of living. It is the only paper in which he deals with demand theory.

When we interviewed Ville’s friends and colleagues, they described him as having his head in the clouds.\footnote{16} Unconcerned by academic reality, Ville was interested only in solving theoretical puzzles in an original way.\footnote{17} According to Bernard Bru, he was “secret, ironic, and benevolent,” had a “reluctance to publish,” and was wary of all types of institutions.\footnote{18} He often flitted from one subject to another, solving a problem and switching to a new one. In fact, his work shows that his interests were incredibly wide ranging. For example, when he was engaged in research on game theory and probability, he gave the first geometric proof of the Minimax theorem of John von Neumann.\footnote{19} The importance of this contribution was stressed by von Neumann:

This connection may now seem very obvious to someone who first saw the theory after it had obtained its present form. . . . However, this was not at all the aspect of the matter in 1921–38. The theorem and its relation to the theory of convex sets were far from being obvious; witness the following facts:

a) In 1921, and thereafter, Borel surmised the theorem to be false or possibly false.

b) In 1928, I proved the theorem by observing its relation to the theory of fixed points and not yet to that one of convex sets.

\footnote{15. Ville’s dissertation is a critique of Richard von Mises’s (Ludwig’s brother) work on mechanics. Incidentally, it is amazing to know that Richard was a strong defender of methodological empiricism, while Ludwig is one of the main “apriorists” in economics.}
\footnote{16. We interviewed Pierre Bouzitat, Bernard Bru, Pierre Crépel, and Pierre-Yves Jaffray, who were very close to Ville.}
\footnote{17. “The characterization of Jean Ville’s works is their constant originality” [“la caractérisation des travaux de Jean Ville est leur constante originalité”]. Manuscript by Fréchet.}
\footnote{18. Private interview by the authors, November 2004.}
\footnote{19. Borel was impressed by von Neumann’s demonstration. He asked Ville to write up his courses, which gave him the opportunity to present another original proof of this theorem.
c) In 1935, I generalized it (for the purpose of the theory of prices and production) by an even more explicit use of the fixed-point method.

d) It took ten years after my original proof until J. Ville discovered, in 1938, the connection with convex sets.

e) Even now, this connection does not tell the entire, or the simplest story, about the theorem, as the work since 1945 of S. Kakutani, J. Nash, J. Broron, and myself shows. (quoted in Ville 1955, 6–7)

Ville also introduced, as early as 1930, the concept of martingale. But he was never fully recognized, and never promoted himself, as a great innovator, even by French mathematicians working in these fields, despite the importance of his contributions. He also dealt with quantum physics, pure and applied mathematics, and the theory of signals. In each field he provided real and decisive contributions. He also wrote poetry (he was at that time close to André Breton) and had many connections with the Montparnasse artistic community, including a friendship with Amedeo Modigliani.

In fact, he seemed concerned only with the intellectual aspects of problems and not at all by their academic consequences. That there was no evident “continuity” in his works and that he was not at all willing to enhance his academic status, especially in economics, may explain why he did not try to defend and develop his 1946 paper. Like other French economists of the time, Ville did not know the academic world of international economists or the way it was organized and functioned. Furthermore, when Houthakker and Samuelson wrote their respective 1950 papers, Ville did not seem to be concerned and did not read them. The explanation for this is that Ville was no longer interested at that time in demand theory. He was dealing with new theoretical puzzles, namely, the theory of signals.

Jean Ville’s Mathematics

When Ville trained as a mathematician in the 1920s at the ENS, the institution was not yet dominated by the famous Bourbaki school (which began

20. Discussion of Schroedinger’s equation using a new analytic operator.
21. He discovered independently the pseudo-inverses in matrix theory but did not publish on it. He also worked on functional spaces in pure mathematics, game theory, information theory, and probability.
22. In this domain, some of Ville’s contributions were “classified,” that is, protected as military secrets.
in 1934). After the end of the nineteenth century, the ENS became the center of the French mathematical school (instead of the Ecole Polytechnique, which dominated the early nineteenth century), with such well-known students as Gaston Darboux, Jean Picard, Paul Painlevé, Jacques Hadamard, Elie Cartan, René-Louis Baire, Émile Borel (Ville’s teacher), and Henri Lebesgue (Henri Poincaré was accepted at the ENS but chose to enter Polytechnique). At that time, both pure and applied mathematics were taught at the ENS, which may explain the particular interest Ville took in many domains of applied mathematics (in Vienna he began a thesis on functional analysis but soon became much more interested in probability). During his whole career, he seemed to prefer to give short proofs of difficult theorems rather than to widen his analysis of problems to provide greater mathematical generality.

In his proof of integrability conditions, he used the theory of Pfaff forms, which had been extensively developed by Darboux (1882, 1898) and by Cartan in his 1922–27 lectures at the Sorbonne (2001, chap. 2). Indeed, Ville did not give the precise reference to the theorem of integrability from Pfaff forms he used in his proof (1946, 38n). This theorem, known as the Darboux theorem, allows a Pfaff form to be written, which is defined by the total differential of total expenditure: \( d\omega = \sum_{i} p_i dq_i \) in terms of a set of independent variables \( z, y, x \) under one of the forms \( dz - \sum_{i} y_i dx_i \) or \( \sum_{i} p_i dq_i \), the number of which depends on the dimension of the Pfaff form. In Ville’s paper, the difference between the two sets of variables, \((p, q)\) and \((z, y, x)\), is not clear. Ville’s proof defines the set of variables \( z, y, x \) in order to create a cycle for total expenditure \( \omega \) (this definition may not be possible for variables \( p, q \) because of possible interdependences between variables in \( p, q \)). Thus it appears that Ville’s mathematical tools belonged to the contemporary mathematics used by the French school at the beginning of the twentieth century. More interesting, this mathematics is related to research on non-Euclidean geometry and to the flatness of Riemannian manifolds (corresponding to integrability conditions for Christoffel symbols; see Gardes 2005). These conditions also indicate whether there is a path-dependency on the manifold, which is a problem clearly related to integrability.

In their remarkable contribution, Hurwicz and Richter generalize Ville’s proof, using the Frobenius theorem, which proves the equivalence between the symmetry of the Antonelli matrix (Antonelli’s conditions, called the symmetry budgeter assumption [SBA] by Hurwicz and Richter) and the uniqueness of the solution to a system of partial derivative equations.
The SBA is also equivalent to the absence of Ville cycles. Hurwicz and Richter’s generalization consists both in supposing that the inverse demand functions are $C^k$ (instead of $C^\infty$) and in the inverse implication SBA $\rightarrow$ no Ville cycle. It should be noted that Hurwicz and Richter also use Darboux’s theorem in their proof. Hurwicz and Richter (1979, 612) also give an interesting discussion on applying Ville’s proof to thermodynamics, which shows that Ville indeed provides a profound insight in the integrability debate.

5. Conclusion

The splendid simplicity of Ville’s proof may be thought by economists to cast aside extensive economic debate about the integrability problem, which led to so many controversies and was solved so rapidly by an elementary application of thermodynamic principles using a classic property of Pfaff forms. Ville did not, however, popularize his theorem in English-speaking academic circles. To sum up:

1. Independently of Samuelson and Houthakker, Ville gave a definition of revealed preference theory in his third assumption (see section 1).
2. At the same time, he defined the weak axiom.
3. And he gave a one-page proof of integrability conditions, a proof that is both the first and the best in terms of its axiomatic completeness and simplicity.

The comparison between Ville’s and Houthakker’s proofs provides room for discussing the assumptions necessary to transform the discrete version of a-cyclicity into a continuous version, which needs to make assumptions about the properties of choice sets (since it appears that these axioms postulate the comparability between different bundles of goods and the path-independence of preference relations).

Appendix A: Ville’s Demonstration of the Integrability Conditions

1. Ville considers demand functions to depend on the prices and income of the consumer:

$$q_i = f_i (p_1, \ldots, p_n, s),$$

assumed as homogeneous of degree zero (and continuous).
2. The cost of the quantity vector $q$ at prices $p' = p + dp$, is

$$p'\cdot q = (p + dp)\cdot q = p\cdot q + (p\cdot dq + q\cdot dp + dp\cdot dq) - p\cdot dq - dp\cdot dq$$

$$= s + ds - p\cdot dq - dp\cdot dq$$

$$\approx s + ds - p\cdot dq \text{ at the first order,}$$

then $p'\cdot q < s + ds \iff p\cdot dq > 0$ (1).

(It is important to notice that Ville reasons in terms of strict inequalities. This makes it possible to pose the problem at the first order, using the continuity of the demand functions.)

The first inequality, $p'\cdot q < s + ds$, implies that the vector of quantities of goods chosen at prices $p + dp$ and for an income $s + ds$ is preferred to the vector of initial quantities chosen at prices and for a vector of quantities chosen for the initial prices and income, because it corresponds to a higher cost. This is the principle of revealed preference: the consumer could continue to choose vector $q$ at new prices $p'$, because the cost would have been lower than the cost of $q + dq$. If consumers choose $q + dq$, it shows that they strongly prefer this combination of consumption, this new vector being more expensive than the initial one. We obtain with equivalence (1):

$$p\cdot dq > 0 \iff q(p + dp, s + ds) > q(p, s).$$

Therefore, the sign of $p\cdot dq$ indicates the preference between $q(p, s)$ and $q(p + dp, s + ds)$.

3. Ville’s fourth assumption stipulates that there is no cycle along a closed contour, which is equivalent to saying that $p\cdot dq$ cannot constantly increase if the standard of living does not continuously increase, implying that $p\cdot dq$ has an integrated factor. The demonstration is as follows:

(i) If $p\cdot dq$ does not have an integrated factor, it is possible to put it in one of two Pfaff forms:

$$d\omega = dz - y\cdot dx = dz - \sum_i y_i dx_i,$$

or

$$d\omega = -y\cdot dx$$

23. $V(x_1, x_2, \ldots, x_n)$ being an $n$ vector field ($i = 1, 2, \ldots, n$), the form $V_1\cdot dx_1 + V_2\cdot dx_2 + \ldots + V_n\cdot dx_n$ is called a Pfaff form (Pfaff forms were extensively studied by Elie Cartan, a French mathematician probably well known to Ville). When the $V_i$ are the coordinates of the gradient of a differentiable function on an open set of $V^n$, then the Pfaff form is the total differential of this function.
with a number $h$ of variables $y_i$ and $x_i$ that are less or equal to the number $n$ of goods. The new variables are in a number, $2h + 1$ or $2h$, less than or equal to the number $n$ of variables $q_i$. Ville notes that the first form is in general the case for an odd number of goods, the second for an even number.

(ii) He defines the closed outline as the following:

$$
x_i = \cos \varphi \\
y_i = \sin \varphi \\
z = - \cos \varphi + \frac{1}{3} \cos^3 \varphi.
$$

The vector $(x, y, z)$ describes the outline when $\varphi$ goes from 0 to $2\pi$, and then $d\varpi = (h + \sin \varphi) \sin^2 \varphi \, d\varphi$ for the first form (with $h > 1 \Rightarrow h + \sin \varphi > 0$), or $h \sin^2 \varphi \cdot d\varphi$ for the second; both keep a constant sign all along the closed outline, which contradicts the fourth assumption. Writing the expression in a Pfaff form is impossible, so $p \cdot dq$ has an integrated factor.

**Appendix B: A Synthetic Figure**

Semi-definiteness of the Antonelli matrix

Symmetry of the Antonelli matrix

Integrability

(1) Under continuity conditions
References


