Inter-sectorial Knowledge Diffusion and Scale Effects

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Abstract

Knowledge diffusion, both within and across sectors, shapes the pools of knowledge used by R&D activities. Exploiting the formalization of Salop (1979), we propose a fully endogenous Schumpeterian model explicitly considering that each innovation influences a more or less significant range of sectors. Besides encompassing most models of the related literature, this framework enables us to reassess the issue of scale effects, showing that most frameworks developed to remove scale effects amount to assuming no inter-sectorial diffusion of knowledge, and that, it is possible to eliminate scale effects while maintaining the effects of public policies, but without suppressing knowledge diffusion.

Keywords: Endogenous Growth, Scale Effects, Knowledge Accumulation, Knowledge Diffusion, Knowledge Spillovers.

JEL Classification: O31, O33, O41
1 Introduction

Technological progress plays a major part in the striking phenomenon of long-run growth. This process of knowledge accumulation has been formalized by the endogenous growth theory based on innovation through two complementary paradigms. The first one considers that growth is driven by a diversification of the variety of intermediate goods, as in Romer (1990), for instance. The second one, developed notably in Grossman & Helpman (1991) or in Aghion & Howitt (1992), introduces quality improving innovations based on stochastic and sequential R&D races. The key-stone of knowledge accumulation lies in the presence of the externality triggered by the non rivalry property of knowledge, commonly referred to as “knowledge spillovers”: knowledge produced in any given sector can potentially spill from this sector over other sectors. In other words, once knowledge has been generated by a research and development (R&D) activity, it diffuses across others R&D activities. The significance of the interactions between sectors (i.e. inter-sectorial knowledge spillovers) has universally been underlined. In particular, several empirical studies stress that R&D performed in one sector may produce positive spillovers effects in other sectors (see, for instance, Griliches, 1992 and 1995, or Hall, Mairesse & Mohnen, 2010). As stated in Hall, Mairesse & Mohnen (2010), “such spillovers are all the more likely and significant as the sender and the receiver are closely related”. Moreover, as argued by Hall (2004), “it is safe to say that without diffusion, innovation would have little social or economic impact. In the study of innovation, the word diffusion is commonly used to describe the process by which individuals and firms in a society/economy adopt a new technology, or replace an older technology with a newer”.

In the standard growth literature, the intensity of knowledge spillovers varies considerably from one framework to the other. In Romer (1990), or in Jones (1995b), all the knowledge accumulated so far in the economy is used to produce new knowledge. Similarly, in Aghion & Howitt (1992), in Young (1998), in Howitt (1999), or in Segerstrom (2000), spillovers depend on the knowledge level of the frontier firms. In other words, in those models, knowledge spillovers are implicitly assumed to be global. To some extend, this theory implicitly takes into account the generally agreed fact that new pieces of knowledge “diffuse gradually, through a process in which one sector gets ideas from the research and experience of others”\(^1\). In contrast, in Grossman & Helpman (1991), in Segerstrom (1998), in Peretto (1999), or in Aghion & Howitt (2009), for example, they have been assumed to be intra-sectorial only (i.e. knowledge is sector-specific). In Peretto (1998), Dinopoulos & Thompson (1998), or Li (2000, 2003), spillovers depend on the average knowledge in the economy. In all of these models, innovations are produced drawing previously created knowledge from a pool of knowledge which is arbitrarily given.

In spite of the fact that knowledge spillovers are ubiquitous in modern growth literature, their formalization has not really been investigated. Notable exceptions are the models of Li (2000 and 2002) and Peretto & Smulders (2002). Li emphasizes the role of knowledge spillovers and shows that semi-endogenous growth theory is more general than fully endogenous growth theory in that the latter relies on knife-edge conditions. The model of Peretto & Smulders (2002) is based on the concept of spillovers networks. It focuses on knowledge spillovers within firms and industries and on the role of market structure in shaping them. In particular, the authors introduce a concept of “technological distance” to examine the interaction between firms.

In the present framework, we consider that the knowledge produced in each sector can potentially influence the R&D activity of any other sector. In direct line with the above empirical statements, we explicitly introduce knowledge diffusion, both within and across sectors, in a canonnic fully endogenous “Schumpeterian” growth model. More precisely, we expressly assume that each piece of knowledge produced in any given sector can diffuse with more or less intensity in the economy. As in Peretto & Smulders, we can define a technological distance. As expected, this distance is decreasing with the average intensity of diffusion of knowledge within the whole economy.

Depending on its extent, an innovation can thus potentially influence the creation of knowledge in sectors that are more or less technologically distant. Firstly, this allows us to propose a more accurate formalization of the aforementioned knowledge spillover externality. Secondly, it enables us to shed a light on the mechanism by which the pools of knowledge in which R&D activities draw from in order to produce new knowledge are shaped.

\(^1\)Aghion & Howitt (1998, chapter 3).
Besides knowledge spillovers, the nonrivalry property of knowledge has another significant implication. Because a piece of knowledge can be used infinitely without any additional cost once its production cost has been incurred, technologies using knowledge as an input (both final good and knowledge production functions) exhibit increasing returns to scale. As stated by Jones (2005), nonrivalry of knowledge is at the source of a non desirable property in the seminal endogenous growth models: scale effects. Indeed, the models of Romer (1990), Grossman & Helpman (1991), Aghion & Howitt (1992) all have in common the counterfactual property predicting that the economy’s long-run per capita growth rate increases in its size, measured by the level of population. Numerous papers have tackled, in various ways, the matter of scale effects within growth models. Jones (1995a) has been the first to point out this problem. In particular, he argues that the presence of the scale effects property - stating that the larger the economy is, the stronger growth will be - does not match with empirics. The specialists all agree on the fact that this property is undesirable since this prediction is strongly inconsistent with twentieth century observed stylized facts. Furthermore, if population grows at a positive and constant rate, then the economy’s per capita growth rate increases exponentially over time and eventually becomes infinite in the steady-state. Therefore, the presence of scale effects raises both an empirical and a theoretical problem.

The issue of scale effects in growth models has been reviewed in a large body of theoretical and empirical literature. However, to the best of our knowledge, it has not been investigated in the light of knowledge diffusion. Nevertheless, because nonrivalry and diffusion of knowledge, on the one hand, and nonrivalry and scale effects, on the other, are intrinsically linked, it appears natural to reconsider the issue of scale effects under this new perspective. A purpose of the paper consists in showing that knowledge diffusion is a key point to analyze the scale effects property.

Jones (1995b) removes the scale effects property from Romer (1990)’s variety-based model. Kortum (1997) or Segerstrom (1998), among others, also provide scale-invariant growth models, in which the long-run growth is proportional to the exogenous rate of population growth. These approaches are based on the notion of “diminishing technological opportunities” and gave birth to the “semi-endogenous growth” literature which exhibits the theoretical link between the growth rate of the economy and the growth rate of the population. In this range of models, scale effects are still present in the determination of the variables levels but no longer of their growth rates. Yet, in this theory, economic policies (especially R&D subsidies) turn to have an impact only on the levels of economic variables, not on the long-run growth rates; moreover, in the absence of positive population growth, the growth rate of the economy is nil.

Contrasting with this growth theory, an alternative range of literature, that one often refers to as “endogenous growth without scale effects theory”, appeared through the impulse of Aghion & Howitt (1998), Dinopoulos & Thompson (1998), Peretto (1998), Young (1998), Howitt (1999) or Peretto & Smulders (2002), among others. These “fully endogenous Schumpeterian” growth models restore the effect of economic policies on long-term growth, without displaying the scale effects property, which is eliminated through a “variety expansion mechanism” which introduces an increasing R&D difficulty. As stated by Dinopoulos & Sener (2007), “horizontal product differentiation takes the form of variety accumulation and removes the scale-effects property from these models [...]. Vertical product differentiation takes the form of quality improvements or process innovations and generates endogenous long-run growth”. In this class of double differentiation models, sectors proliferation dilutes R&D effort in a larger number of different sectors, thus dissipating its effect on the overall rate of productivity growth. It appears clearly that the
latter theory relies on two assumptions. Firstly, the scale of the economy, as measured by the level of its population, has an impact on the number of sectors. More precisely, an increase in population results in a proportionate increase in the variety of goods (i.e. in the number of sectors, assumed to be a measure of the R&D difficulty) in the economy. Regarding this first key point, there are several ways to justify this assumption, whether it is empirically or theoretically as, for example, in Laincz & Peretto (2006) and in Dinopoulos & Sener (2007), respectively. The second assumption is more intricate; it relates to the nature of the pools of knowledge used by each sector that is generally considered in these fully endogenous growth models. Peretto & Smulders (2002) accurately describe it as follows: “R&D productivity depends on some measure of accumulated public knowledge that is independent of the number of firms and hence of the scale of the economy. This independence may stem from the assumption that (a) spillovers across firms are absent (Peretto, 1999), that (b) spillovers depend on average knowledge (Smulders & Van de Klundert, 1995; Peretto, 1998; Dinopoulos & Thompson, 1998), or that (c) spillovers depend on the knowledge of the most advanced firm (Young, 1998; Aghion & Howitt, 1998; Howitt, 1999). All these models have the property that a large economy replicates the structure of a small economy. [...] Moreover, although they allow for spillovers, all these models assume that a larger number of firms undertaking independent R&D projects does not support a larger aggregate stock of public knowledge. 'The implicit assumption is that all public knowledge is replicated'”.

In this paper, we go further in this direction, arguing that scale effects are eliminated by implicitly wiping out inter-sectorial knowledge diffusion. We show that, in order to eliminate this property while maintaining the effects of public policies on the growth rate of the economy, these models introduce a normalization assumption, which implies that the knowledge production function in any given sector intrinsically depends on the level of knowledge in this sector only. This feature seems at odds with the common view on how knowledge springs into existence, since it somehow eludes part of its fundamental non rivalry property.

Is it possible to eliminate scale effects while maintaining inter-sectorial knowledge diffusion? The present paper notably tackles this central issue; in particular, we show that, under some reasonable assumptions, the answer is positive. The model we present provides a foundation for a general class of Schumpeterian growth models, without changing the main insights of the standard literature. As stated above, the new feature of our framework lies in that a general formalization of knowledge accumulation is derived from commonly agreed assumptions. Furthermore, we introduce explicitly the role played by knowledge diffusion in the constitution of the pools of knowledge used by R&D activities to create new knowledge.

Regarding the creation of knowledge, our analysis is in the direct line of the seminal works of Romer, Grossman & Helpman, or Aghion & Howitt, even though we present a formalization of knowledge accumulation which is slightly different from theirs. The process of knowledge production is decomposed in two steps: the occurrence of new ideas and the consecutive knowledge increase. We keep the commonly shared assumption of stochastic arrival of innovations: as in Grossman & Helpman (1991), or Aghion & Howitt (1992), we assume a Poisson arrival rate of new ideas, which depends positively on the research effort. Regarding knowledge increases, we consider that, in each sector, they depend positively on a given pool of knowledge in which R&D activities draw from to produce new knowledge. From those two basic assumptions, we derive the law of motion of knowledge in each sector. The latter generalizes many of the various laws of motion considered in the literature. Indeed, making assumptions on the functional relationships and on the nature of the pools previously mentioned enables us to encompass most of the standard growth models, whether fully endogenous -with or without scale effects-, or semi-endogenous. For instance, one can obtain the laws of motion of Romer (1990) or of Jones (1995b).

Besides the fact that we propose a quite general formalization of the process of knowledge accumulation, we explicit a mechanism through which the pools of knowledge evoked above are shaped, introducing knowledge diffusion. In this respect, we exploit the formalization of circular product differentiation model of Salop (1979) to take into account the fact that knowledge can either be specific to the sector in which it has been created or diffuse variously among R&D sectors, ranging from local to global diffusion. Each point of the clockwise oriented circle corresponds
to an intermediate sector, having its own R&D activities producing innovations. Allowing for miscellaneous scopes of diffusion of knowledge has a critical aftermath on the composition of the pool of knowledge used by R&D activities. We consider three possible types of knowledge diffusion. Once an innovation has occurred, the diffusion of the inherent knowledge can be intra-sectorial (sector specific innovation), narrow or wide. The limit case of wide inter-sectorial diffusion is global diffusion, in which knowledge is used by all the sectors in the economy.

It is noteworthy that the seminal frameworks proposed in the beginning of the nineties appear as implicit particular cases of knowledge diffusion. Indeed, a specificity models à la Grossman & Helpman consists in that each sector makes use of the knowledge produced within this sector only, i.e. it has his own specific pool of knowledge. On the contrary, models à la Romer (as well as models à la Aghion & Howitt to a certain extent) assume that each sector uses the same pool of knowledge, which consists of the whole disposable knowledge. In other words, the former consider that there is no inter-sectorial knowledge diffusion (solely intra-sectorial knowledge diffusion), and the latter assume global inter-sectorial knowledge diffusion. Whereas the fully endogenous Schumpeterian literature generally considers one of these two polar cases of knowledge diffusion, our framework allows us to analyse all intermediate cases in between. Moreover, various scopes of diffusion can coexist within a single model.

We characterize the set of decentralized equilibria as functions of two types of public tools: subsidies (or, potentially, taxes) to R&D activities and subsidies to each intermediate good demand. The first basic result exhibits a new determinant of growth: the growth rate of the economy is increasing in the intensity of knowledge diffusion within the economy. The second standard result, which is in line with the fully endogenous growth theory, consists in that public policies have an impact on the equilibrium growth rate. The main results of the paper relate to the presence of the counterfactual property of scale effects. In order to quantify their significance, we introduce a measure, which allows us to clarify why it is possible to eliminate scale effects while maintaining knowledge diffusion in a fully endogenous Schumpeterian growth model.

The paper is organized as follows. In section 2, we present the model. We pay a particular attention to the description of the formalization of the way knowledge accumulates and diffuses among R&D activities. Accordingly, we explain how the pools of knowledge spring to existence. Furthermore, we describe the decentralized economy. Section 3 focuses on showing how the standard theory fits into our model, starting with seminal endogenous growth with scale effects. Then we propose an explanation, based on knowledge diffusion analysis, of how scale effects have been removed in fully endogenous Schumpeterian models. In section 4, we study the interaction between the size of the economy, inter-sectorial knowledge diffusion and scale effects. Considering several cases, we show in particular that the significance of scale effects depends on the speed at which the scope of knowledge diffusion increases with the size of the economy. We present three characteristic cases. Firstly, we consider the elementary case in which the scope of knowledge diffusion does not increase with the size of the economy. In this case, scale effects are nil. Secondly, the scope of knowledge diffusion is assumed to spread, but less quickly than the size of the economy: scale effects decrease over time and asymptotically vanish. Finally, we introduce the fundamental case of global diffusion, which can somehow be related to the issue of general purpose technologies. Accordingly, the scope of knowledge diffusion increases at the same speed as the size of the economy. In this limit case, even if scale effects remain, it can be argued that they are not significant since the probability of occurrence of general purpose technologies is low. We conclude in Section 5.
2 Model

This section presents the fundamentals of the economy studied in the paper. We consider a continuous-time Schumpeterian growth model, in which innovations can diffuse, with more or less intensity, within the sectors R&D activities. In this respect, we exploit the formalization of a circular product differentiation model of Salop (1979). The key ingredients of the model developed in this paper lie in the formalization of knowledge accumulation and diffusion within the economy.

2.1 Knowledge Accumulation

There is a continuum $\Omega_t$, of measure $N_t$, of intermediate sectors uniformly distributed on a clock-wise oriented circle. Each intermediate sector $\omega$, $\omega \in \Omega_t$, is characterized by an intermediate good $\omega$, produced in quantity $x_\omega$, and by a level of knowledge $\chi_\omega$. It has its own R&D activity which is dedicated to the creation of innovations. Each innovation successively increases the amount of knowledge inherent in this sector. Accordingly, we define the whole disposable knowledge in the economy, at each date $t$, as:

$$K_t = \int_{\Omega_t} \chi_{\omega t} \, d\omega$$  \hspace{1cm} (1)

The mechanism at the source of the creation of knowledge relies on two core assumptions. Firstly, the innovation process is uncertain. We assume (Assumption A1) that, if $I_{\omega t}$ is the amount of labor devoted to R&D at date $t$ in any intermediate sector $\omega$, $\omega \in \Omega_t$, to move on to the next quality of intermediate good $\omega$, innovations occur randomly with a Poisson arrival rate $\lambda(I_{\omega t})$, where $\lambda(.)$ is an increasing function of class $C^1$.

In the standard Schumpeterian growth theory, there is a quality ladder for each intermediate good: each innovation takes the intermediate good quality up by one rung on this ladder. Each R&D activity creates new knowledge (innovations) making use of previously created knowledge. The second assumption (Assumption A2) formalizes this idea by considering that, in order to produce new knowledge, each sector $\omega$ draws from a specific pool of knowledge, denoted $P_\omega$. Formally, for any intermediate good $\omega$, $\omega \in \Omega_t$, if an innovation occurs at date $t$, the increase in knowledge, $\Delta \chi_{\omega t}$, (i.e. the quality improvement) depends on the current size of the pool of knowledge in which this sector’s R&D activity draws from: $\Delta \chi_{\omega t} = \sigma(P_\omega)$, $\forall \omega \in \Omega_t$, where $\sigma(.)$ is an increasing function. From those two assumptions, one derives the law of motion of the average knowledge inherent in any sector $\omega$, as expressed in Proposition 1 below.

**Proposition 1:** Under assumptions A1 and A2, on average, the law of motion of the knowledge characterizing any intermediate sector $\omega$ is:

$$\dot{\chi}_{\omega t} = \lambda(I_{\omega t}) \sigma(P_\omega) \, \forall \omega \in \Omega_t$$

The proof is as follows. Consider any given sector $\omega$, $\omega \in \Omega_t$, and a time interval $(t, t + \Delta t)$. At date $t$, the knowledge in this sector is $\chi_{\omega t}$. Let $k$, $k \in N$, be the number of innovations that occur during the interval $(t, t + \Delta t)$. Under assumptions A1 and A2, the knowledge at date $t + \Delta t$, $\chi_{\omega t + \Delta t}$, is a random variable taking the values $\{\chi_{\omega t} + k \sigma(P_\omega)\}_{k \in N}$ with associated probabilities

$$\frac{(f_{t+\Delta t}^{\lambda(I_{\omega u})} du)^k}{k!} e^{-f_{t+\Delta t}^{\lambda(I_{\omega u})} du} \sigma(P_\omega)$$.  \hspace{1cm} \forall k \in N$

Therefore, the expected level of knowledge at date $t + \Delta t$ is:

$$E[\chi_{\omega t + \Delta t}] = \sum_{k=0}^{\infty} \frac{(f_{t+\Delta t}^{\lambda(I_{\omega u})} du)^k}{k!} e^{-f_{t+\Delta t}^{\lambda(I_{\omega u})} du} \sigma(P_\omega)$$

$$= \chi_{\omega t} \sum_{k=0}^{\infty} \frac{(f_{t+\Delta t}^{\lambda(I_{\omega u})} du)^k}{k!} \sigma(P_\omega) \left(\int_{t}^{t+\Delta t} \lambda(I_{\omega u}) \, du\right) \sum_{k=1}^{\infty} \frac{(f_{t+\Delta t}^{\lambda(I_{\omega u})} du)^{k-1}}{(k-1)!} e^{-f_{t+\Delta t}^{\lambda(I_{\omega u})} du}$$
The MacLaurin series $\sum_{k=0}^{K} \frac{(f_{t+\Delta}^t \lambda(l_{u_t})du)^k}{k!}$ converges to $e^{f_{t+\Delta}^t \lambda(l_{u_t})du}$ as $K \to \infty$. Thus, one gets:

$$\mathbb{E} [\chi_{\omega t + \Delta \tau}] = \left[ \chi_{\omega t} e^{f_{t+\Delta}^t \lambda(l_{u_t})du} + \sigma \left( P_t^\omega \right) \left( \int_{t}^{t+\Delta t} \lambda(l_{u_t}) du \right) e^{f_{t+\Delta}^t \lambda(l_{u_t})du} \right] e^{-f_{t+\Delta}^t \lambda(l_{u_t})du}$$

$$\Leftrightarrow \mathbb{E} [\chi_{\omega t + \Delta \tau}] = \chi_{\omega t} + \left( \int_{t}^{t+\Delta t} \lambda(l_{u_t}) du \right) \sigma \left( P_t^\omega \right)$$

Let $\Lambda_{\omega u}$ denote a primitive of $\lambda(l_{u_t})$ with respect to the time variable $u$. Finally, rewriting the previous expression, we can exhibit the Newton’s difference quotients of $\mathbb{E} [\chi_{\omega t}]$ and of $\Lambda_{\omega t}$:

$$\frac{\mathbb{E} [\chi_{\omega t + \Delta \tau}] - \chi_{\omega t}}{\Delta \tau} = \frac{\Lambda_{\omega t + \Delta \tau} - \Lambda_{\omega t}}{\Delta \tau} \sigma \left( P_t^\omega \right)$$

Letting $\Delta \tau$ tend to zero, one gets $\frac{\partial \mathbb{E} [\chi_{\omega t}]}{\partial \tau} = \mathbb{E} [\chi_{\omega t}] = \lambda(l_{\omega t}) \sigma \left( P_t^\omega \right)$. This proves that the expected knowledge in any intermediate sector $\omega$, $\omega \in \Omega_t$, is a differentiable function of time. Its derivative gives the law of motion of the expected knowledge as given in Proposition 1 above, in which the expectation operator is dropped to simplify notations. $\square$

The law of motion of knowledge derived in Proposition 1 encompasses many of the ones assumed in the standard theory, depending on the specifications of the functions $\lambda(\cdot)$ and $\sigma(\cdot)$, and on the choice of the pools $P_t^\omega$. Two illustrations are presented in the corollary below.

**Corollary:**

1. Assume that $\lambda(l_{\omega t}) = \lambda_{\omega t} (\lambda > 0)$, $\sigma \left( P_t^\omega \right) = \sigma P_t^{\omega \Phi} (\sigma > 0, \Phi < 1)$, and $P_t^\omega = K_t, \forall \omega \in \Omega_t$ (the pool of knowledge used by each sector $\omega$ is the whole knowledge accumulated so far). One gets $\dot{\chi}_{\omega t} = \lambda_\omega l_{\omega t} K_t^{\Phi}$, $\forall \omega \in \Omega_t$. Furthermore, assuming $N_t = N$, one has $\Omega_t = \Omega, \forall t$. Therefore, summing on $\Omega$, one obtains:

$$K_t^\omega = \lambda_\omega l_{\omega t} R_t^{\omega \Phi},$$

where $R_t^{\omega \Phi} = \int_{\Omega} l_{\omega t} \ d\omega$ is the overall amount of labor dedicated to research in the economy.

This law of motion is formally identical to the one proposed by Jones (1995b) in the semi-endogenous growth theory.

2. Assume that $\lambda(l_{\omega t}) = \lambda_{\omega t} (\lambda > 0)$, and that $\sigma \left( P_t^\omega \right) = \sigma P_t^\omega (\sigma > 0)$ (i.e., when an innovation occurs in sector $\omega$, the increase in knowledge $\Delta \chi_{\omega t}$ is proportional to the current size of the pool of knowledge used by this sector). The law of motion of the knowledge characterizing any intermediate sector $\omega$ is:

$$\dot{\chi}_{\omega t} = \lambda_\omega l_{\omega t} P_t^\omega, \forall \omega \in \Omega_t$$

The law of motion (2) is quite general. Indeed, setting arbitrarily the pools $P_t^\omega$, one can obtain several laws of knowledge accumulation commonly used in the fully endogenous-growth “Schumpeterian” theory. However, as it will be developed in Subsection 2.2 below, we are not going to set arbitrarily the pools of knowledge used in each sector; we precise the mechanism through which these pools are formed.

In this paper, we focus only on fully endogenous-growth theory. In this respect, the law of knowledge accumulation considered from now on is given by (2).

### 2.2 Knowledge Diffusion and Pools of Knowledge

Now that we have sketched the way knowledge accumulates, in this subsection we explicit how the pools of knowledge are shaped, taking into account knowledge diffusion. Knowledge inherent in a given sector can diffuse variously among R&D sectors, ranging from local to global diffusion.
Therefore, the constitution of the pools relies on the influence that each R&D activity has on the others. Each sector is simultaneously a sender and a receiver of knowledge: in the following, the index $\omega$, $\omega \in \Omega_t$, is used to point out a sector from which knowledge $\chi_h$ diffuses (the sender); the index $\omega$, $\omega \in \Omega_t$, is used to point out the sector that may potentially use this knowledge (the potential receiver).

For any R&D activity $\omega$, $\omega \in \Omega_t$, the disposable pool of knowledge, $\mathcal{P}_t^{\omega}$, is composed of the knowledge produced in this sector so far, as well as of knowledge diffused from other sectors. In this respect, we propose a formalization which explicates the process of knowledge diffusion. We assume that, when an innovation occurs in any sector, it can either be specific to the R&D sector in which it has been created ("sector specific innovation"), or, it can diffuse locally to R&D activities ("narrow innovation"), or, finally, it can diffuse more broadly on a larger set of R&D activities ("wide innovation").

Let us define the scope of diffusion of an innovation as the measure of the subset of sectors of $\Omega_t$ which are able to use the knowledge inherent in this innovation, or, in other words, as the measure of the neighborhood of diffusion of this knowledge. Formally, when an innovation occurs in any intermediate sector $h$, $h \in \Omega_t$, its scope of diffusion is a random variable $\theta$ which can take three values$^8$: 0, $\theta$ or $\bar{\theta}_t$, respectively with probabilities $p_0$, $p_\theta$ and $p_{\bar{\theta}_t}$, where $1 < \theta < \bar{\theta}_t \leq N_t$ and $p_0 + p_\theta + p_{\bar{\theta}_t} = 1$. Accordingly, the expected scope of diffusion of innovations is:

$$E[\theta_t] = p_\theta \theta + p_{\bar{\theta}_t} \bar{\theta}_t$$  \hspace{1cm} (3)

It is a measure of the intensity of inter-sectorial knowledge spillovers, which is comprised between zero (no inter-sectorial spillovers / only intra-sectorial spillovers) and $N_t$ (only global inter-sectorial spillovers). Moreover, we assume that knowledge diffuses symmetrically over $\Omega_t$. Thus, the neighborhoods of diffusion of knowledge $\chi_h$ inherent in sector $h$ (i.e. the subset of sectors of $\Omega_t$ which are able to use the knowledge inherent in sector $h$), $h \in \Omega_t$, in the case of narrow innovations and of wide innovations are respectively: $\Omega_h^\theta \equiv [h - \theta/2; h + \theta/2]$ and $\Omega_h^\bar{\theta}_t \equiv [h - \bar{\theta}_t/2; h + \bar{\theta}_t/2]$, where $\Omega_h^\theta \subseteq \Omega_h^\bar{\theta}_t \subseteq \Omega_t$. In the case of sector specific innovations ($\theta = 0$), there is no sector using knowledge $\chi_h$ besides sector $h$ itself. Accordingly, one gets the expression of $\mathcal{P}_t^\omega$, given in Proposition 2 below.

**Proposition 2:** At each date $t$, in any intermediate sector $\omega$, $\omega \in \Omega_t$, the expected pool of knowledge used by the R&D activity is:

$$\mathcal{P}_t^\omega = (1 - p_0 - p_{\bar{\theta}_t}) \chi_{\omega t} + p_\theta \int_{\Omega_h^\theta} \chi_{ht} dh + p_{\bar{\theta}_t} \int_{\Omega_h^\bar{\theta}_t} \chi_{ht} dh, \forall \omega \in \Omega_t$$  \hspace{1cm} (4)

The proof is the following.

Let us refer to an innovation in any given sector $h$ as to "innovation $h$". As stated above, any innovation $h$, $h \in \Omega_t$, consists either in a sector specific innovation with probability $p_0 = 1 - p_\theta - p_{\bar{\theta}_t}$, in a narrow innovation with probability $p_\theta$, or in a wide innovation with probability $p_{\bar{\theta}_t}$. In the first case, the only sector specific innovation $h$, $\forall h \in \Omega_t$, reaching the location of R&D activity $\omega$ is innovation $\omega$ itself. The corresponding amount of knowledge received by R&D activity $\omega$ is then $\chi_{\omega t}$.

In the second case, solely narrow innovations $h$ which are located in the nearby neighborhood $\Omega_h^\theta$ can get to R&D activity $\omega$. The amount of knowledge inherent in narrow innovations which can be used by R&D activity $\omega$ is then $\int_{\Omega_h^\theta} \chi_{ht} dh$.

Finally, all wide innovations $h$ which are located in the neighborhood $\Omega_h^\bar{\theta}_t$ reach R&D activity $\omega$. Thus, the amount of knowledge inherent in wide innovations and received by R&D activity $\omega$ is $\int_{\Omega_h^\bar{\theta}_t} \chi_{ht} dh$. Consequently, the expected total amount of knowledge used by any R&D activity $\omega$, $\omega \in \Omega_t$, is given by the expression (4) above. $\square$

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$^8$In order to simplify the model, we assume that, without loss of generality, $\bar{\theta}$ is independent of time, whereas $\bar{\theta}_t$ potentially depends on time. Indeed, we will consider different cases in which $\bar{\theta}_t$ is independent of time or not, depending on the nature of the diffusion of wide innovations. We name “global innovation”, the limit case of a wide innovation which diffuses to the whole economy: $\bar{\theta}_t = N_t$. One generally refers to innovations that diffuse globally as to “general purpose technologies".
To keep the analysis tractable, we will often make the usual assumption of symmetry\(^9\): \(\chi_{\omega t} = \chi_t\) and \(l_{\omega t} = l_t\), \(\forall \omega \in \Omega_t\). Accordingly, the pools of knowledge and the laws of accumulation of knowledge in each sector \(\omega\) are:

\[
\mathcal{P}_t^\omega = \mathcal{P}_t = (p_0 + E[\theta]) \chi_t \text{ and } \dot{\chi}_{\omega t} = \lambda \sigma l_t (p_0 + E[\theta]) \chi_t, \forall \omega \in \Omega_t
\]  

(5)

Introducing the fact that innovations are differentiated with respect to their scope of diffusion generalizes the standard innovation-based endogenous growth theory: it allows us to consider a large collection of models, according to the set of parameters \((p_0, p_n, p_W, \theta, \xi)\) chosen. Firstly, the scope of diffusion of any innovation is comprised between zero (sector specific innovations) and \(N_t\) (global innovations). Secondly, innovations with different scopes of diffusion can coexist within a model.

The frameworks originally proposed by Grossman & Helpman (1991) and by Aghion & Howitt (1992) both correspond to two particular cases of this framework. Indeed, consider that the number of sectors is constant \((N_t = N)\). Assuming that any innovation is sector specific \((i.e.\) that there is only intra-sectorial diffusion of knowledge), one obtains a model close to Grossman & Helpman. Considering that any innovation is global \((i.e.\) that there is both intra and global inter-sectorial diffusion), one gets a slightly modified version of Aghion & Howitt (1992). The corollary below presents formally under which assumptions these particular cases are obtained and the resulting main characteristics.

**Corollary - Particular cases:**

Assume \(L_t = L\) and \(N_t = N\). Accordingly, one has \(\Omega_t = \Omega, \forall t\).

1. A growth model à la Grossman & Helpman is obtained for \(p_0 = 1\) \((i.e.\) \(p_n = p_W = 0\)). Accordingly, one one gets:

\[
\Omega^\omega = \Omega^\omega = \{\omega\}, E[\theta] = 0 \text{ and } \mathcal{P}_t^\omega = \chi_{\omega t}
\]

This model, in which knowledge spillovers are only intra-sectorial, is characterized by the following knowledge production function:

\[
\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \chi_{\omega t}, \forall \omega \in \Omega
\]

2. A growth model à la Aghion & Howitt is obtained for \(p_W = 1\) \((i.e.\) \(p_0 = p_n = 0\)) and \(\overline{\theta} = N\). Accordingly, one gets:

\[
\Omega^\omega = \Omega^\omega = \left[ -\frac{N}{2}; \frac{N}{2} \right] = \Omega, E[\theta] = N \text{ and } \mathcal{P}_t^\omega = \int_\Omega \chi_{\omega t} d\omega = \mathcal{K}_t
\]

This model, in which knowledge spillovers are global, is characterized by the following knowledge production function:

\[
\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \mathcal{K}_t, \forall \omega \in \Omega
\]

We will get back on those two particular cases in more detail in Subsection 3.1. It is noteworthy that the expression of the knowledge production (6), which is endogenously derived from assumptions made in a stochastic quality ladders model, leads to a law of motion of the whole disposable knowledge which is formally identical to the knowledge production function initially introduced by Romer (1990). Indeed, differentiating (1) with respect to time yields:

\[
\dot{\mathcal{K}}_t = \int_\Omega \dot{\chi}_{\omega t} d\omega = \lambda \sigma \left( \int_\Omega l_{\omega t} d\omega \right) \mathcal{K}_t \iff \dot{\mathcal{K}}_t = \lambda \sigma L_t^R \mathcal{K}_t,
\]

As mentioned previously, the issue of knowledge spillovers has been tackled in Peretto & Smulders (2002). They consider that the “extent to which a firm can take advantage of the public knowledge created by other firms decreases with the technological distance between the creator

---

and the user of such knowledge\textsuperscript{2}. In their paper, the average technological distance between an arbitrary pair of firms increases with the size of the economy. Using the explicit formalization of knowledge diffusion enables us to propose a similar concept of technological distance, which is also increasing in the size of the economy (as measured by the number of sectors). Furthermore, we argue that the average technological distance should be decreasing in the expected scope of diffusion of innovations. Accordingly, a trivial measure of technological distance within the present framework can be assumed to be:

\[ D_t = \frac{N_t}{E[\theta]_t} \in [1; \infty) \]  

In order to illustrate (7), let us compute it in the two particular cases introduced above. In models \textit{à la} Grossman & Helpman, the technological distance is infinite \((D_t = \infty, \text{ since } E[\theta] = 0)\), meaning that sectors cannot share knowledge. In models \textit{à la} Aghion & Howitt, it is minimal, \((D_t = D = 1, \text{ since } E[\theta] = N)\); here, on the contrary, all sectors benefit from the whole stock of knowledge.

2.3 The Environment

Now that we have presented the way knowledge accumulates, let us complete the presentation of the economy. We denote by \(g_t\), the rate of growth, \(z_t/z_t\), of any variable \(z_t\). Population, \(L_t\), grows at constant rate \(g_t = n, n > 0\). Each household is modelled as a dynastic family which maximizes the discounted utility\textsuperscript{10} \(U = \int_0^\infty L_t u(c_t)e^{-\rho t}dt\), where \(\rho > n\) is the common subjective discount rate and \(u(c_t)\) is the individual instantaneous utility at time \(t\), which is given by\textsuperscript{11} \(u(c_t) = \ln(c_t)\). The initial size of the population, \(L_0\), is normalized to unity. Then, the population of workers in the economy at time \(t\) is \(L_t = e^{nt}\). Intertemporal preferences of the representative household are thus given by:

\[ U = \int_0^\infty \ln(c_t)e^{(n-\rho)t}dt \]

At each date \(t\), each of the \(L_t\) identical households is endowed with one unit of labor that is supplied inelastically. It is used, in quantity \(L_t^Y\), to produce the final good and in R&D activities. Thus, the labor constraint is:

\[ L_t = L_t^Y + \int_{\Omega_1} l_{\omega t} d\omega \]

Besides labor, the production of the final good requires the use of all available intermediate goods, each of which is associated with its own level of knowledge. The final good production technology is:

\[ Y_t = (L_t^Y)^{1-\alpha} \int_{\Omega_1} \chi_{\omega t}(x_{\omega t})^\alpha d\omega , \quad 0 < \alpha < 1, \]  

The final good has two competing uses. Firstly, it is used in the production of intermediate goods along with:

\[ x_{\omega t} = \frac{y_{\omega t}}{\chi_{\omega t}} , \quad \omega \in \Omega_t , \]  

where \(y_{\omega t}\) is the quantity of final good used to produce \(x_{\omega t}\) units of intermediate good \(\omega\). This technology illustrates the increasing complexity in the production of intermediate goods: as the quality of a given intermediate good increases, its production requires more resources. Secondly, it is consumed by the representative household in quantity \(c_t\). One gets the following constraint on the final good market:

\[ Y_t = L_t c_t + \int_{\Omega_1} y_{\omega t} d\omega \]

In the following sections, we study a decentralized economy which is in direct line with the analysis conducted by Aghion & Howitt (1992), in which R&D activities are funded by monopoly

\textsuperscript{10}Barro & Sala-i-Martin (1995, chapter 2) provide more details on this formulation of the households behavior within the context of the Ramsey model of growth. See also Segerstrom (1998).

\textsuperscript{11}The results are robust if one considers a more general C.E.S. instantaneous utility function of parameter \(\varepsilon \in [0; 1]\), \(u(c_t) = \frac{c_t^{1-\varepsilon} - 1}{1-\varepsilon}\).
profits on the sale of intermediate goods embodying the knowledge. We normalize the price of final good to one, and denote respectively by \( w_t, r_t \) and \( q_{\omega t} (\omega \in \Omega_t) \) the wage, the interest rate and the price of intermediate good \( \omega \) at instant \( t \). The final good market, the labor market and the financial market are perfectly competitive. Once invented, an intermediate good can be modified, improved as the result of several steps of innovations. Regarding their markets, we consider Schumpeter’s “creative destruction” mechanism. It involves that, in a given intermediate sector, the firm that succeeds in innovating is granted a patent and can monopolize the intermediate good production and sale until replaced by the next innovator.

There is potentially a divergence between the equilibrium allocation and the first-best optimal one. Indeed, as usual, there are two sources of inefficiency. The first one results from the presence of monopolies on the production and sale of intermediate goods. This distortion can be mitigated by an \textit{ad valorem} subsidy \( \psi, \varphi \in [0; 1] \), on each intermediate good demand. The second externality is triggered by the fact that there is no market for knowledge (i.e. there are knowledge spillovers). It can be corrected by a public tool \( \varphi, \varphi \in \mathbb{R} \), which can consist in a subsidy or in a tax on the profits of R&D activities, depending on whether the R&D effort is sub-optimal or over-optimal. We characterize the set of equilibria as functions of the public tools vector \((\psi, \varphi)\): at each vector \((\psi, \varphi)\) is associated a particular equilibrium. Formally, an equilibrium is represented as time paths of set of prices \( \{(p_{\omega t}, \omega \in \Omega_t), w_t, r_t\}_{t=0}^{\infty} \) and of quantities \( \{(e_t, Y_t, \{x_{\omega t}\}_{\omega \in \Omega_t}, \{L^t_{\omega t}\} \}_{\omega \in \Omega_t}, \{\chi_{\omega t}\}_{\omega \in \Omega_t}, \{K_t\}_{t=0}^{\infty}\} \), such that: the final good market, the labor market and the financial market clear; on each intermediate good market, the incumbent monopolizes the production and sale until replaced by the next innovator; there is free entry on each R&D activity (i.e. the zero profit condition holds for each R&D activity); firms maximize their profits and the representative household maximizes her utility.

In the next sections, in order to shed a new light on the issue of scale effects, we compute the growth rate of per capita consumption as function of \((\psi, \varphi)\); we denote it by \( g_t(\psi, \varphi) \). All the computations as well as the complete characterization of the decentralized economy can be found in Appendix.

### 3 Growth Theory with and without Scale Effects: The Standard Literature

In this section, we show that this model generalizes the standard fully endogenous growth literature. In particular, we recover the main results regarding the scale effects property. Moreover, this unifying framework enables us to shed a new light on the way the standard literature removes this non desirable property. We proceed in two steps. Firstly, we assume that the size of the population and the measure of the set of intermediate goods sectors are constant, and show that the resulting model exhibits scale effects, as well as the seminal models of the literature. Secondly, we introduce population growth, examine how the literature has removed scale effects, and show that the commonly shared assumption introduced by several authors is equivalent to undermine the role of knowledge diffusion.

#### 3.1 Growth Theory with Scale Effects

Let us first assume that the size of the population and the measure of the set of intermediate goods sectors are constant: \( L_t = L \) (i.e. \( n = 0 \)) and \( N_t = N \), for all \( t \). For consistency, we assume that the maximal diffusion of knowledge (i.e. the diffusion of wide innovations) is constant: \( \bar{\theta}_t = \bar{\theta} \), for all \( t \). Hence, from (3), \( \mathbb{E}[\theta] = p_n \bar{\theta} + p_W \bar{\theta} \) is independent of time. The growth rate, expressed as a function of any vector of public tools \((\psi, \varphi)\) is\(^{12}\):

\[
g(\psi, \varphi) = \frac{\lambda \sigma L}{N} \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{2N}{\alpha}}{(1 + \varphi)\alpha + (1 - \psi)} \right) (p_0 + \mathbb{E}[\theta])
\]  

(13)

As seen in (13), both economic policy tools, \( \psi \) and \( \varphi \), have a positive impact on the decentralized equilibrium growth rate of per capita consumption. Moreover, this general class of models

\(^{12}\)Computations of the growth rate are provided in Appendix. See Lemma 1.
exhibits scale effects, since the rate of growth, \( g(\psi, \varphi) \), is increasing in \( L \). Those two results are the ones of the seminal growth literature. Finally, let us underline that the growth rate is increasing in the intensity of inter-sectorial knowledge spillovers, \( E[\theta] \). The model presented in this section thus generalizes the existing literature by explicitly introducing the role of knowledge diffusion. In order to illustrate the general nature of this framework, we get back on the two particular cases evoked above in the corollary of Proposition 2. Models à la Grossman & Helpman (obtained for \( E[\theta] = 0 \)) are characterized by:

\[
\begin{align*}
g_{E[\theta]=0}(\psi, \varphi) &= \frac{\lambda \sigma L}{N} \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{\theta N}{\lambda \sigma L}}{(1 + \varphi)\alpha + (1 - \psi)} \right) 
\end{align*}
\]

Models à la Aghion & Howitt (obtained for \( E[\theta] = N \)) are characterized by:

\[
\begin{align*}
g_{E[\theta]=N}(\psi, \varphi) &= \frac{\lambda \sigma L}{N} \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{\psi N}{\lambda \sigma L}}{(1 + \varphi)\alpha + (1 - \psi)} \right)
\end{align*}
\]

As stated above, both of these types of models exhibit scale effects at any equilibrium. Proposition 3 below summarizes the basic result of the subsection.

**Proposition 3:** Assume \( L_1 = L \) and \( N_1 = N \). The model exhibits the scale effects property for any \( E[\theta] \).

The issue of the presence of scale effects has been tackled in various ways, both empirically and theoretically. In particular, two major types of methodologies have been developed to remove this unwanted property, giving birth to two branches of growth models: semi-endogenous and fully endogenous. Successive reviews have given rise to a debate regarding the respective relevancy of using one or the other type of frameworks. As mentioned above in this paper, we focus on the “endogenous growth without scale-effects“. In the next subsection, we investigate how scale effects have been cancelled from fully endogenous Schumpeterian growth models.

### 3.2 Fully Endogenous Growth Theory without Scale Effects

Let us now consider population growth and focus on how the seminal models of fully endogenous growth have cancelled scale effects.

The models of endogenous growth without scale-effects introduce constant population growth (\( g_L = n > 0 \)) while maintaining the effects of public policies. They all have in common to suppose that the number of sectors is proportional to employment, i.e. \( N_t = \gamma L_t \), where \( \gamma \) is a constant. In other words, the number of sectors increases as the population level does so. As stated by Laincz & Peretto (2006), this relation “might induce one to conclude that this class of models requires another “knife-edge” condition in that one needs to assume that the number of firms is exactly proportional to population”. They add that there are several ways to justify this first assumption. Their main argument in favor of this relation is empirical: according to their data, the number of establishments is proportional to employment. Another defense for this relationship echoes to Young (1998)’s insight that, as population grows, the proliferation of the intermediate sectors reduces the efficiency of R&D activities in improving the quality of an existing product because the R&D effort is diluted in more sectors. Here, the scale effects property is removed through a “variety expansion mechanism”.

---

13. Li (2000), for instance, argues that semi-endogenous growth theory is more general than endogenous growth theory. Ha & Howitt (2007) maintain that fully endogenous growth is more accurate. They argue that empirical evidences are more supportive of fully endogenous Schumpeterian growth theory than they are of semi-endogenous growth theory.

14. This is underlined by Jones (1999), Laincz & Peretto (2006), or Dinopoulos & Sener (2007), among others.

15. This can be introduced endogenously as in e.g. Howitt (1999) or Segerstrom (2000). It has also been set up in a more straightforward scheme as exposed, for instance, in Aghion & Howitt (2009, chapter 4, section 4.4). As an illustration, let us adapt the methodology they propose in our framework: at date \( t \), the probability of inventing a new intermediate good is a linear function of the population size, \( L_t \). Unlike in Howitt (1999), there are no R&D expenditure here. Moreover, we suppose that an exogenous fraction \( \xi \) of intermediate goods becomes obsolete and vanishes at each date \( t \). The variation of the number of sectors at each date \( t \) is given by \( \dot{N}_t = \kappa L_t - \xi N_t \), where
In the case of a framework à la Grossman & Helpman (1991), that is in models in which there is no inter-sectorial diffusion of knowledge, or in other words, in which the pool of knowledge used in each sector is solely the knowledge produced in this sector, the condition \( N_t = \gamma L_t \) is sufficient to remove the scale effects property (see e.g. Young, 1998). We will get back to this issue below, in Proposition 5, in which this result is proved.

In models in which the pool of knowledge used in each sector comprise all knowledge accumulated so far in the whole economy, this first condition is not sufficient. One needs to add a second condition so as to remove the scale effects property while maintaining the effects of public policies (see, for instance, the models proposed by Dinopoulos & Thompson, 1998, Howitt, 1999, or Segerstrom, 2000). As Emphasized in the overviews provided by Jones (1999), Laincz & Peretto (2006), or Dinopoulos & Sener (2007), in all of these models, the analysis is focused on the sector level rather than on the economy level. More precisely, we show that those formalizations lean on a normalization of the pool of disposable knowledge by the number of sectors, and hence equivalently by the level of the population. The argument commonly put forward to justify this normalization by the size of the economy is as follows. Given that the number of sectors is a measure of R&D difficulty, dividing the pools of knowledge by the size of the economy is a way to account for the fact that higher R&D difficultly implies that the same amount of R&D resources generates a lower production of knowledge. In other words, in order to introduce decreasing-account for the fact that higher R&D difficultly implies that the same amount of R&D resources generates a lower production of knowledge. In other words, in order to introduce decreasing-

Those frameworks have the downside of reducing to models in which inter-sectorial knowledge spillovers eventually vanish. Formally, we show that this assumption of normalization is indeed equivalent to wiping out knowledge diffusion, as underlined in Proposition 4.

**Proposition 4:** The normalization introduced in the standard “endogenous growth without scale-effects” is equivalent to assuming \( E[\theta] = 0, \forall t \) (or, equivalently, \( p_0 = 1 \)), in the present framework.

The proof is the following.

Using (1) and (14), one easily sees that the expression of pools of knowledge usually considered in the literature is \( P^\omega_t = \frac{1}{N_t} \int_0^{N_t} \chi h_t dh = \frac{N_t}{\gamma L_t} \chi L_t \). Now, consider the usual symmetric case, in which \( \chi = \chi, \forall \omega \in \Omega_t \). Accordingly, one has \( P^\omega_t = N_t \chi, \forall \omega \in \Omega_t \). In the framework we propose, the pools are given by (5): \( P^\omega_t = (p_0 + E[\theta] \chi L_t, \forall \omega \in \Omega_t \). Identifying those two expressions yields \( P_0 + E[\theta] \chi L_t = 1 \Leftrightarrow p_0 + p_0 \theta + p_W \theta_l = 1 \Leftrightarrow p_t (\theta - 1) + p_W (\theta_l - 1) = 0 \).

Since \( 1 < \theta < \theta_l \), the last expression holds if and only if \( p_n = p_W = 0 \Leftrightarrow E[\theta] = 0, \forall t \). □

Consider the particular case of the model we have introduced, in which one assumes \( g_{Lt} = n \geq 0, N_t = \gamma L_t \), and \( E[\theta] = 0 \). The pools of knowledge and the laws of accumulation of knowledge \( \kappa \) and \( \xi \) are positive parameters, and where \( L_t = e^{nt} \). The solution of this non-homogeneous first-order linear differential equation is:

\[
N_t = \frac{\kappa}{n + \xi} \left( e^{nt} - e^{-\xi t} \right), \forall t \Leftrightarrow N_t \frac{N_t}{L_t} = \frac{\kappa}{n + \xi} \frac{e^{nt} - e^{-\xi t}}{e^{nt}} = \frac{\kappa}{n + \xi} \left( 1 - e^{-\xi (n + \xi) t} \right), \forall t
\]

Therefore, the ratio length of the list of intermediate sectors over population level will eventually stabilize at a steady-state value, \( (N_t/L_t)^{ss} = \kappa/(n + \xi) \equiv \gamma \). Indeed, \( (N_t/L_t) \) converges to \( \kappa/(n + \xi) \) as \( t \to \infty \), because \( \xi + n > 0 \).

16Note that one could have alternatively referred to equations (7) and (9) in Jones (1999), to equations (13) and (14) in Dinopoulos & Sener (2007), to equation (5) in Ha & Howitt (2007), or to the framework used in Aghion & Howitt (2009, chapter 4).
in each sector, in the symmetric case, are respectively:
\[ P_t = \chi_{\omega t} = \chi_t \quad \text{and} \quad \dot{\chi}_{\omega t} = \dot{\chi}_t = \lambda \sigma_t \chi_t, \forall \omega \in \Omega_t \]  
(15)

The growth rate is\(^\text{17}\):
\[ g(\psi, \varphi) = \frac{\lambda \sigma}{\gamma} \left( \frac{(1 + \varphi)(1 - (1 - \psi)\frac{\gamma}{\alpha})}{(1 + \varphi)(1 - \psi)} \right) + n \]  
(16)

Like in usual models of the endogenous growth without scale effects literature, there are no scale effects in this particular case of our framework\(^\text{18}\), and the effects of public policies are maintained, as seen in the expression of the growth rate (16). However, as in this standard literature, one considers here that there are no inter-sectorial spillovers (i.e. that \( \mathbb{E}[\theta] = 0 \)), or, equivalently, that the technological distance, \( \mathcal{D}_t \), is infinite (see equation (7)). In other words, in this framework, the pool of knowledge used in each sector comprises only the knowledge accumulated in this one, as seen in (15).

Since the normalization assumption introduced to remove scale effects is equivalent to assume that the intensity of inter-sectorial knowledge spillovers is nil, it implicitly suppresses knowledge diffusion across sectors. In other words, the models originally considering global inter-sectorial diffusion of knowledge (polar case à la Aghion & Howitt) boil down to models in which there is only intra-sectorial diffusion of knowledge (polar case à la Grossman & Helpman). In accordance with the formalization we have introduced, Proposition 5 below summarizes the main insights of the endogenous growth theory without scale effects.

**Proposition 5:** Assume \( g_L = n > 0 \) and \( N_t = \gamma L_t \). If \( \mathbb{E}[\theta], = 0, \forall t, \) (i.e. if there is no inter-sectorial diffusion of knowledge), both policy tools, \( \psi \) and \( \varphi \), have a positive impact on the decentralized equilibrium growth rate and the model does not exhibit the scale effects property.

Concluding this section, one has to emphasize that, in order to remove the scale effects property, the standard endogenous growth without scale effects theory implicitly eliminates knowledge diffusion across sectors. Ignoring this fundamental property of non-rivalry of knowledge seems to ignore the common view on how knowledge springs into existence. Indeed, it is generally agreed that new pieces of knowledge “diffuse gradually, through a process in which one sector gets ideas from the research and experience of others”\(^\text{19}\).

Is it still possible to remove the non desirable property of scale effects while maintaining the effects of public policies, but without annihilating knowledge diffusion? We tackle this question in the next section.

4 Size of the Economy, Knowledge Diffusion, and Scale Effects

In this section, we study the consequences of various assumptions on the scope of diffusion of knowledge on three issues: i) the effects of public policies, ii) the presence of scale effects, iii) the existence of a non explosive growth rate. Throughout the section, the size of the economy, measured by the level of population, \( L_t \), increases at the constant rate \( n > 0 \). Moreover, we keep the commonly shared assumption that the number of sectors is proportional to employment, \( N_t = \gamma L_t \).

We study the property of scale effects, that is the impact of the size of an economy on its growth rate. More precisely, we investigate the link between knowledge diffusion and the presence of scale effects. Recall that, within the model, knowledge diffusion is characterized by its scope, \( \mathbb{E}[\theta], = p_n \theta + p_w \theta, \forall t \). Moreover, since \( \theta, \in (\theta, N_t] \), this scope is influenced by the size of the economy.

\(^{17}\)This growth rate is obtained setting \( \mathbb{E}[\theta] = 0 \) in the expression (37) of the growth rate exhibited in Lemma 2 of the Appendix.

\(^{18}\)The model exhibits weak scale effects; however, the growth rate of the economy is strictly positive even if the population growth rate is nil. For more details on weak scale effects, see, for instance, Jones (2005).

\(^{19}\)Aghion & Howitt (1998, chapter 3).
We focus on the usual symmetric case studied in the standard Schumpeterian approach, in which \( \chi_\omega = \chi_I \) and \( L_\omega = l_t, \forall \omega \in \Omega_t \). Therefore, in each sector, the resulting pools and laws of accumulation of knowledge are \( P_t^I = P_t = (p_0 + E[\theta^I_t]) \chi_I, \forall \omega \in \Omega_t \) and \( \chi_t = \lambda \sigma l_t (p_0 + E[\theta^I_t]) \chi_t \), respectively. Finally, given a vector of public policies \( (\psi, \varphi) \), the equilibrium in the decentralized economy is characterized by the following growth rate of per capita consumption:\(^{20}\)

\[
g_t (\psi, \varphi) = \frac{\lambda \sigma}{\gamma} \left( \frac{(1 + \varphi) \alpha - (1 - \psi) \Sigma^L}{(1 + \varphi) \alpha + 1 - \psi} \right) (p_0 + E[\theta^I_t]) + n \tag{17}
\]

From (17), we deduce the two following Lemmas.

**Lemma 1:** Both policy tools, \( \psi \) and \( \varphi \), have a positive impact on the equilibrium growth rate, \( g_t (\psi, \varphi) \), for any \( E[\theta^I_t] \).

**Lemma 2:** Knowledge diffusion, measured by \( E[\theta^I_t] \), has a positive impact on the rate of growth, \( g_t (\psi, \varphi) \), for any vector of public tools \( (\psi, \varphi) \).

The model potentially exhibits the scale effects property since the growth rate depends positively on \( E[\theta^I_t] \), which itself depends on the size of the economy, measured equivalently by \( L_t \) or \( N_t \). In order to apprehend with more accuracy the impact of knowledge diffusion, let us introduce a measure of scale effects:

\[
S_t = \frac{\partial g_t (\psi, \varphi)}{\partial L_t} = \left\{ \begin{array}{ll} \frac{\lambda \sigma}{\gamma} \left( \frac{(1 + \varphi) \alpha - (1 - \psi) \Sigma^L}{(1 + \varphi) \alpha + 1 - \psi} \right) p_W \frac{\partial \theta^I_t}{\partial L_t}, & \text{for all } p_W > 0 \\
0, & \text{for } p_W = 0 \end{array} \right. \tag{18}
\]

Recall that, as seen in Proposition 5 above, in the case in which inter-sectorial knowledge diffusion is removed (i.e. when one assumes that \( E[\theta^I_t] = 0, \forall t \)), the scale effects property is eliminated. This result is revisited here. Indeed, in this limit case, one assumes that \( p_W = 0 \). Hence, the measure of scale effects, \( S_t \), is nil.

In this section, we assume that \( E[\theta^I_t] \neq 0, \forall t \). We proceed in three steps, each of which consists in making an assumption on \( \bar{\theta}_t \), the scope of diffusion of wide innovations. We start by considering a simple case in which it is assumed to be independent of time: \( \bar{\theta}_t = \bar{\theta}, \forall t \). This enables us to propose a scale invariant growth model, which explicitly takes into account inter-sectorial knowledge diffusion. Then, we consider that, as the economy expands, \( \bar{\theta}_t \) increases, but less quickly. A particular case of this set up consists in assuming that \( \bar{\theta}_t \) is bounded above. This framework restores scale effects; however, they are limited. Finally, we show that, even when one considers the possibility of global diffusion of knowledge \( (\bar{\theta} = N_t) \), scale effects are tenuous if the probability of innovations to diffuse globally \( (p_W) \), is low. The main results are summarized in Propositions 6, 7, and 8, below.

### 4.1 Constant Knowledge Diffusion: \( \bar{\theta}_t = \bar{\theta} \)

Let us first consider the simple case in which \( \bar{\theta}_t \) is independent of time. Consequently, the scope of diffusion of knowledge is also time invariant: \( E[\theta^I_t] = E[\theta] = p_W \bar{\theta} + p_W \bar{\theta}, \forall t \). Accordingly, as seen in (17), there are no scale effects in this set up. In other words, their measure, \( S_t \), is nil, as seen in (18). Moreover, contrary to the case summarized in Proposition 5 above, this model does not eliminate the assumption of inter-sectorial knowledge diffusion. Therefore, it is possible to completely eliminate scale effects while maintaining the effects of public policies on the growth rate, and without wiping out knowledge. These results are summarized in Proposition 6.

**Proposition 6:** Assume \( g_{L_t} = n > 0 \) and \( N_t = \gamma L_t \). If \( \bar{\theta}_t = \bar{\theta}, \forall t \), the model does not exhibit the scale effects property (i.e. \( S_t = 0 \)) for any \( E[\theta] \). Moreover, the rate of growth, \( g(\psi, \varphi) \), is constant over time.

We thus provide a scale-invariant model which allows for population growth and knowledge diffusion. Unlike in the standard theory, scale effects are removed while preserving inter-sectorial

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\(^{20}\) Computations of the growth rate are provided in Appendix. See Lemma 2.
knowledge spillovers. However, the scope of diffusion of wide innovations, $\overline{\theta}_t$, and thus, the intensity of knowledge spillovers, $E[\theta]_t$, have been assumed to be time invariant. Due to the non-rivalry property of knowledge, it perhaps seems unreasonable to suppose that, as the economy expands, the scope of knowledge diffusion remains constant. In the next subsection, we relax this assumption: we introduce the fact that, as the size of the economy (measured equivalently by $L_t$ or $N_t$) increases, the scope of diffusion of wide innovations spreads.

4.2 Extending Knowledge Diffusion: $\overline{\theta}_t = \overline{\theta}(N_t)$

Because knowledge is a non rival good, it appears natural to think that the scope of knowledge diffusion increases with the size of the economy. In this subsection, we consider that the scope of diffusion of wide innovations, $\overline{\theta}_t$, increases over time. Since $N_t = \gamma L_t = \gamma e^{\alpha t}$, $\forall t$, this is equivalent to assume that $\overline{\theta}_t$ is an increasing function of $N_t$: $\overline{\theta}_t = \overline{\theta}(N_t)$, with $\overline{\theta}'(N_t) > 0$. Accordingly, the intensity of knowledge spillovers, $E[\theta]_t = p_W \overline{\theta} + p_W \overline{\theta}(N_t)$, increases as the economy extends. As seen in (17), scale effects are restored. Therefore, we potentially face two problems. On the one hand, the presence of the scale effects property has been proved to be strongly at odds with empirics. On the other, this property involves a theoretical problem: if population grows at a positive and constant rate, then the economy’s growth rate increases exponentially over time and becomes infinite.

In what follows, we show that, under reasonable assumptions, none of these issues is relevant within our framework. In this respect, we examine two cases. Firstly, the scope of diffusion of wide innovations, $\overline{\theta}_t$, grows, but not at as fast as the size of the economy. A particular case of this framework would be to consider that it is bounded from above. Secondly, we assume that $\overline{\theta}_t$ is an increasing and strictly concave differentiable function of class $C^2$. The concavity assumption can be justified by arguing that the expansion of the economy goes along with an increasing complexity which curbs down the speed at which the scope of diffusion of knowledge spreads. The results obtained under those assumptions are presented in Proposition 7 below.

Proposition 7: Assume $g_{L_t} = n > 0$, $N_t = \gamma L_t$, and $\overline{\theta}_t = \overline{\theta}(N_t)$, where $\overline{\theta}(\cdot) \in C^2$, with $\overline{\theta}'(N_t) > 0$ and $\overline{\theta}''(N_t) < 0$.

i) For any $p_W > 0$, the model exhibits the scale effects property:

$$\delta_t = \lambda \sigma \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{\varphi}{\lambda}}{(1 + \varphi)\alpha + 1 - \psi} \right) p_W \overline{\theta}'(N_t) > 0$$

ii) For any $p_W > 0$, the measure of scale effects is decreasing over time: $\delta_t < 0, \forall t$.

iii) Moreover, if $\lim_{N_t \to \infty} \overline{\theta}'(N_t) = 0$, scale effects asymptotically vanish: $\lim_{t \to \infty} \delta_t = 0$.

Corollary: Without any extra assumption on $\overline{\theta}(\cdot)$, the intensity of knowledge spillovers, $E[\theta]_t$, and thus the growth rate, $g_t(\psi, \varphi)$, increase over time and can potentially become infinite. However, if $\overline{\theta}(\cdot)$ is bounded above (i.e., if, under the assumptions of Proposition 7, $\lim_{N_t \to \infty} \overline{\theta}'(N_t) = \overline{\theta}$, where $\overline{\theta}$ is finite), $g_t(\psi, \varphi)$ is increasing over time but bounded above.

Despite the fact that scale effects are restored, the model is not really inconsistent with empirics. Indeed, as shown in Proposition 7, as the size of the economy increases, scale effects progressively lessen and eventually disappear. As regards the theoretical problem involved by the presence of scale effects, if the extra assumption of bounded diffusion is added, the economy’s growth rate increases but does not become infinite.
The remaining issue needed to be examine is the case in which the scope of diffusion of wide innovations, \( \bar{\theta}_t \), grows at the same rate as the size of the economy.

### 4.2.2 Global Diffusion: \( \bar{\theta}_t = N_t \)

Let us now take a look at the limit case in which we introduce global diffusion of knowledge: some innovations reach all the sectors in the economy. In other words, we consider the possible arrival of a type of general purpose technologies. Formally, we assume that the scope of diffusion of wide innovations (which occur with probability \( p_W \)) is \( \bar{\theta}_t = N_t \). Thus, the intensity of inter-sectorial knowledge spillovers is \( E[\theta]_t = p_0 \bar{\theta} + p_W N_t = p_0 \bar{\theta} + p_W \gamma L_t \).

We get the following result.

**Proposition 8:** Assume \( g_{L_t} = n > 0 \), \( N_t = \gamma L_t \), and \( \bar{\theta}_t = N_t \). For any \( p_W > 0 \), the measure of scale effects is constant over time:

\[
S_t = \lambda \sigma \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\alpha^2}{(1 + \varphi)\alpha + 1 - \psi} \right) p_W
\]

This measure is decreasing when \( p_W \) decreases, and \( \lim_{p_W \to 0} S_t = 0 \).

The presence of global innovations implies scale effects. The significance of scale effects in this case clearly depends on the probability of emergence of these global innovations, \( p_W \). It is empirically reasonable to assume that this type of general purpose technologies (GPT) are quite rare in the large mass of discoveries. In the present framework, this would correspond to assuming that \( p_W \) is small with respect to the probabilities of specific and narrow innovations, \( p_0 + p_n \). To sum up, even in the case in which there are scale effects due to the presence of global diffusion of knowledge, their impact on growth is limited since the probability of occurrence of general purpose technologies is low.

In Proposition 9 below, using the concept of technological defined in (7), we come back on the three cases examined in section 4.

**Proposition 9:** Assume \( g_{L_t} = n > 0 \) and \( N_t = \gamma L_t \).

i) If \( \bar{\theta}_t = \bar{\theta}, \forall t \), then \( E[\theta]_t = \bar{\theta} \), \( D_t \) is an increasing function of time and \( \lim_{t \to \infty} D_t = \infty \).

ii) If \( \bar{\theta}_t = \bar{\theta} (N_t) \), then \( E[\theta]_t = p_0 \bar{\theta} + p_W \bar{\theta} (N_t) \). If, moreover, \( \bar{\theta} (.) \in C^2 \), with \( \bar{\theta} (N_t) > 0 \) and \( \bar{\theta}'' (N_t) < 0 \), \( D_t \) is an increasing function of time. Finally, if \( \lim_{N_t \to \infty} \bar{\theta} (N_t) = \bar{\theta} \) (where \( \bar{\theta} \) is finite), then \( \lim_{t \to \infty} D_t = \infty \).

iii) If \( \bar{\theta}_t = N_t \), then \( E[\theta]_t = p_0 \bar{\theta} + p_W N_t \). Thus, \( D_t \) is an increasing function of time and \( \lim_{t \to \infty} D_t = 1/p_W \). Furthermore, \( \lim_{p_W \to 0} (\lim_{t \to \infty} D_t) = \infty \).

It appears clearly in this proposition that, under several assumptions, in each of the three cases investigated, the technological distance becomes infinite. Under exactly the same assumptions, we have shown in Propositions 6, 7, and 8, that scale effects disappear. More precisely, as seen in Propositions 7 and 9(ii), we have shown that, asymptotically, scale effects dissipate and the technological distance becomes infinite, respectively. Here, as in Peretto & Smulders (2002), “dilution of public knowledge causes the scale effect to vanish asymptotically. The mechanism behind this result is increasing technological distance.” Moreover, they suggest that “general purpose technologies might [thus] decrease technological distance”. As seen in Proposition 9-iii), we propose a model which exhibits this result: the larger the probability of occurrence of GPT \( (p_W) \), the weaker the technological distance. Moreover, in Proposition 8, as stated above, we emphasize the link between the presence of scale effects and the occurrence of GPT.

Consequently, one could think that the absence of scale effects is directly linked with the fact that the technological distance is infinite. In fact, this is not accurate. Indeed, as shown in Proposition 6, it is possible to have simultaneously a positive scope of diffusion, a finite technological distance, and no scale effects.
5 Conclusion

Knowledge diffusion within and across sectors shapes the pools of knowledge in which R&D activities draw from to innovate and, thus, plays a crucial role in the level of disposable knowledge in an economy. This paper developed a fully endogenous Schumpeterian growth model which explicitly takes into account this phenomenon of knowledge diffusion to reassess the already well debated issue of scale effects.

In order to consider that each innovation can diffuse with more or less magnitude within the economy and, therefore, influence a more or less significant range of R&D activities, we combine the formalization of a circular product differentiation model of Salop (1979) with the main insights of the standard models of endogenous growth based on innovation. The presented framework appears to encompass most models of the related literature, whether one considers endogenous growth (both with and without scale effects), or semi-endogenous growth theory. Firstly, we derived a general and very tractable law of accumulation of knowledge. Secondly, introducing explicitly knowledge diffusion allowed us to explain how the pools of knowledge arise. It turns out that most models considered so far by the standard literature can be obtained from the formalization introduced in this paper by making specific assumptions related to the accumulation of knowledge and to its diffusion.

This rather general framework first allowed us to shed a new light on the formalization used in the standard fully endogenous growth models to remove the non desirable scale effects property. In particular, we showed that the frameworks generally developed to avoid the presence of scale effects amount to considering that there is no inter-sectorial diffusion of knowledge (in other words, the pool used in each sector is limited to the knowledge which is specific to this sector).

Besides giving an alternative view on the endogenous growth theory without scale effects, the crux of this paper tackles the issue of determining whether it is possible to eliminate scale effects while maintaining the effects of public policies, but without suppressing knowledge diffusion. We gave three answers to this question. Firstly, we started by providing a simple framework for which the answer is positive. More precisely, we proposed a fully endogenous Schumpeterian model with inter-sectorial knowledge diffusion, which does not exhibit scale effects if the maximum diffusion of knowledge is assumed to be independent of the size of the economy. This latter assumption can be relaxed, considering that the maximum diffusion is extending as the number of sectors increases. In this more reasonable setting, two sub-cases have been investigated. In the first one, the maximum diffusion spreads more slowly than the economy. Scale effects recur; however, their impact on growth decreases and vanishes asymptotically. In the second one, we considered the limit case in which we allowed some innovations to diffuse to the whole economy (i.e. allowed for “global diffusion”); then, the maximum diffusion spreads at the same speed as the economy. Scale effects remain, but their impact is tenuous. Indeed, we showed that the significance of scale effects depends positively on the probability of occurrence of global innovations, which can be assumed to be rather low since such innovations are quite rare in the whole mass of discoveries.
6 Appendix: Decentralized Economy

Agents Behavior

The representative household maximizes her intertemporal utility given by (8) subject to her budget constraint: \( \dot{b}_t = w_t + r_t b_t - c_t - n b_t - T_t / L_t \), where \( b_t \) denotes the per capita financial asset and \( T_t \) is a lump-sum tax charged by the government in order to finance public policies. This yields the usual Keynes-Ramsey condition:

\[
 r_t = g_{c_t} + \rho 
\]  \hspace{1cm} (19)

In the final sector, the competitive firm maximizes its profit given by:

\[
 \pi_t^Y = (L_t^Y)^{1-\alpha} \int_\Omega \chi_{\omega t}(x_{\omega t})^\alpha d\omega - w_t L_t^Y - \int_\Omega (1-\psi) q_{\omega t} x_{\omega t} d\omega 
\]

The first-order conditions yield:

\[
 w_t = (1-\alpha) \frac{Y_t}{L_t} \quad \text{and} \quad q_{\omega t} = \frac{\alpha(L_t^Y(1-\alpha) \chi_{\omega t}(x_{\omega t})^{\alpha-1}}{1-\psi}, \forall \omega \in \Omega_t \]  \hspace{1cm} (20)

In each intermediate good sector \( \omega, \omega \in \Omega_t \), the incumbent monopoly maximizes its profit \( \pi_t^{x_\omega} = q_{\omega t} x_{\omega t} - y_{\omega t} = (q_{\omega t} - \chi_{\omega t}) x_{\omega t} \), where the demand for intermediate good \( \omega \), is given by (20). After maximization, one obtains the standard symmetric use of intermediate goods in the final good production and the mark-up on the price of intermediate goods:

\[
 x_{\omega t} = x_t = \left( \frac{\alpha^2}{1-\psi} \right)^{\frac{1}{1-\psi}} L_t^Y \quad \text{and} \quad q_{\omega t} = \frac{\chi_{\omega t}}{\alpha}, \forall \omega \in \Omega_t \]  \hspace{1cm} (21)

Together with the definition of the whole disposable knowledge in the economy (1), (21) allows us to rewrite the final good production function (10), the wage expression given in (20), and the instantaneous monopoly profit on the sale of each intermediate good \( \omega \), respectively as:

\[
 Y_t = \left( \frac{\alpha^2}{1-\psi} \right)^{\frac{1}{1-\psi}} L_t^Y K_t \quad \text{and} \quad w_t = (1-\alpha) \left( \frac{\alpha^2}{1-\psi} \right)^{\frac{1}{1-\psi}} K_t \quad \text{and} \quad \pi_t^{x_\omega} = \left( \frac{1-\alpha}{\alpha} \right) \left( \frac{\alpha^2}{1-\psi} \right)^{\frac{1}{1-\psi}} L_t^Y \chi_{\omega t}, \forall \omega \in \Omega_t \]  \hspace{1cm} (22)

Given (11), the final good resource constraint (12) becomes \( Y_t = L_t C_t + [\alpha^2/(1-\psi)] \left( \frac{1}{1-\psi} \right)^{\frac{1}{1-\psi}} L_t^Y K_t \). Dividing both sides by \( Y_t \) and using the previous expression of \( Y_t \) gives: \( L_t C_t / Y_t = 1 - \alpha^2/(1-\psi) \). One obtains:

\[
 g_{Y_t} = g_{C_t} + n \]  \hspace{1cm} (23)

Moreover, log-differentiating with respect to time the expression of the final good production function given in (22), gives:

\[
 g_{Y_t} = g_{L_t^Y} + g_{K_t} \]  \hspace{1cm} (24)

Let us now consider any R&D activity \( \omega, \omega \in \Omega_t \), and derive the innovators’ arbitrage condition. Given the governmental intervention on behalf of research activities, the incumbent innovator, having successfully innovated at date \( t \), receives, at any date \( \tau > t \), the net profit \( \pi_t^{x_\omega} = (1+\psi)\pi_t^{x_\omega} \) with probability \( e^{-\int_t^\tau \lambda_{\omega u} du} \) (i.e. provided that there is no innovation upgrading intermediate good \( \omega \) between \( t \) and \( \tau \)). The sum of the present values of the incumbent’s expected net profits on the sale of intermediate good \( \omega \), at date \( t \), is therefore:

\[
 \tilde{\Pi}_t^\omega = \int_t^\infty \tilde{\pi}_\tau^{x_\omega} e^{-\int_t^\tau (r_u + \lambda_{\omega u}) du} d\tau, 
\]

Differentiating this expression with respect to time gives the standard arbitrage condition in each R&D activity \( \omega \):

\[
 r_t + \lambda_{\omega t} = \frac{\tilde{\Pi}_t^\omega}{\tilde{\Pi}_t^\omega} + \tilde{\pi}_t^{x_\omega} \quad \forall \omega \in \Omega_t \]  \hspace{1cm} (25)
The free-entry condition\(^{21}\) in each R&D activity \(\omega\) is \(w_t = \lambda \tilde{\Pi}_t^\omega\), where \(\lambda \tilde{\Pi}_t^\omega\) is the expected revenue when one unit of labor is invested in R&D\(^{22}\), and \(w_t\) is the cost of one unit of labor (given in (22)). This condition gives \(\tilde{\Pi}_t^\omega = \tilde{\Pi}_t = (1 - \alpha) \left[ \alpha^2/(1 - \psi) \right]^{\frac{1}{\alpha}} K_t / \lambda, \ \forall \omega \in \Omega_t\). Consequently, one has \(\tilde{\Pi}_t^\omega / \tilde{\Pi}_t^\omega = g_{\chi_t}\), and \(\tilde{\Pi}_t^\omega / \tilde{\Pi}_t = (1 + \varphi) \lambda \omega / (1 - \psi) K_t\), \(\forall \omega \in \Omega_t\). Replacing in (25), the arbitrage condition can be rewritten as:

\[
r_t + \lambda \omega t = g_{\chi_t} + \frac{(1 + \varphi) \lambda \omega l_t^\chi t}{(1 - \psi)} K_t, \ \forall \omega \in \Omega_t
\]

(26)

**Symmetric Equilibrium**

As in the standard literature, in order to keep the model tractable, we assume \(\chi_{\omega t} = \chi_t\) and \(l_{\omega t} = l_t, \ \forall \omega \in \Omega_t\). That is \(L_t^\chi = N_t l_t\) and \(K_t = N_t \chi_t\). Therefore, the growth rate of the whole disposable knowledge in the economy is given by:

\[
g_{\chi_t} = g_{\chi_t} + n
\]

(27)

Furthermore, in any sector \(\omega\), the pool of knowledge used by R&D activities and the law of motion of knowledge become \(P_t^\omega = P_t = (p_0 + p_n \theta + p_v \theta_t) \chi_t = (p_0 + E[\theta_t]) \chi_t\) and \(\chi_{\omega t} = \chi_t = \lambda l_t (p_0 + E[\theta_t]) \chi_t\), respectively. Therefore, one has the following knowledge growth rates in each sector \(\omega\):

\[
g_{\chi_t} = \lambda \sigma (p_0 + E[\theta_t]) l_t, \ \forall \omega \in \Omega_t
\]

(28)

Finally, we can rewrite (26), the arbitrage condition in any R&D activity \(\omega, \ \omega \in \Omega_t\), as:

\[
r_t + \lambda l_t = \lambda \sigma (p_0 + E[\theta_t]) l_t + n + \frac{(1 + \varphi) \lambda \omega l_t^\chi t}{(1 - \psi)} K_t
\]

(29)

The equilibrium quantities, growth rates and prices are characterized by equations (9), (19), (21), (22), (23), (24), (27), (28) and (29):

\[
\begin{align*}
L_t &= L_t^\chi + N_t l_t \\
r_t &= g_{ci} + \rho \\
x_{\omega t} &= x_t = \left( \frac{\alpha^2}{1 - \psi} \right)^{\frac{1}{\alpha}} L_t^\chi \\
q_{\omega t} &= \frac{\chi_{\omega t}}{\chi_t}, \ \forall \omega \in \Omega_t \\
Y_t &= \left( \frac{\alpha^2}{1 - \psi} \right)^{\frac{1}{\alpha}} L_t^\chi K_t, \ \forall \omega \in \Omega_t \\
gv_t &= g_{ci} + n \\
g\omega_t &= g_{L_t} + g_{\chi_t} \\
g_{\chi_t} &= g_{\chi_t} + n \\
g_{\chi_{\omega t}} &= g_{\chi_t} = \lambda \sigma (p_0 + E[\theta_t]) l_t, \ \forall \omega \in \Omega_t \\
r_t + \lambda l_t &= \lambda \sigma (p_0 + E[\theta_t]) l_t + n + \frac{(1 + \varphi) \lambda \omega l_t^\chi t}{(1 - \psi) N_t}
\end{align*}
\]

(30)

Using (12) and (19), one gets:

\[
g_{ci} + \rho + \lambda l_t = \lambda \sigma (p_0 + E[\theta_t]) l_t + n + \frac{(1 + \varphi) \lambda \omega l_t^\chi t}{(1 - \psi) N_t}
\]

(31)

Moreover, from (15), (16), (17) and (18), one has:

\[
g_{ci} + n = g_{L_t} + g_{\chi t} + n \Rightarrow g_{ci} = g_{L_t} + \lambda \sigma (p_0 + E[\theta_t]) l_t
\]

(32)

Combining (31) and (32) gives:

\[
g_{L_t} + \rho + \lambda l_t = n + \frac{(1 + \varphi) \lambda \omega l_t^\chi t}{(1 - \psi) N_t}
\]

\(^{21}\)Equivalently, one could consider here the zero profit condition.

\(^{22}\)Indeed, innovations in sector \(\omega\) are assumed to occur along with a Poisson arrival rate of \(\lambda \omega l_t\); for one unit of labor is invested in R&D activity \(\omega\), the probability to obtain one innovation at date \(t\) is thus \(\lambda\). Moreover, its value, taking into account the R&D public policy, is \(\tilde{\Pi}_t^\omega\).
Finally, using the labor constraint (11), and rearranging the terms, one gets the following differential equation in \( L_t^Y \):

\[
g_L^Y - \frac{\lambda}{N_t} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] L_t^Y = n - \frac{\lambda L_t}{N_t}
\]

Let us now successively consider two cases:

**Case 1 - No Population Growth:** \( L_t = L \) (i.e. \( n = 0 \)), \( N_t = N \), and \( \bar{g}_t = \bar{g}, \forall t \).

**Case 2 - Population Growth:** \( g_{L_t} = n > 0 \), \( N_t = \gamma L_t \), and \( \bar{g}_t = \bar{g}(N_t), \forall t \).

### 6.1 No Population Growth

In this first case, we consider the seminal framework in which the population level and the number of sectors are assumed to be constant. Moreover, we suppose that the maximum diffusion of knowledge is independent of time. Formally, we assume that \( L_t = L \) (i.e. \( n = 0 \)), \( N_t = N \), and \( \bar{g}_t = \bar{g}, \forall t \). Accordingly, one has \( \Omega_t = \Omega, \forall t \), and (33) writes:

\[
g_L^Y = \frac{\lambda}{N} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] L_t^Y = - \left( \frac{\lambda L}{N} + \rho \right)
\]

In order to solve this differential equation, we use a variable substitution. Let \( X_t = 1/L_t^Y \), log-differentiation with respect to time on gets \( g_{X_t} = -g_L^Y \). Substituting into (34) gives the following first-order linear differential equation:

\[
-g_{X_t} = \frac{\lambda}{N} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] \frac{1}{X_t} = - \left( \frac{\lambda L}{N} + \rho \right) \leftrightarrow \dot{X}_t = \left( \frac{\lambda L}{N} + \rho \right) X_t = - \frac{\lambda}{N} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right]
\]

Its solution is:

\[
X_t = e^{\left( \frac{\lambda L}{N} + \rho \right) t} \left( X_0 - \frac{N}{\lambda L + \rho N} \frac{\lambda}{1 + \frac{1 + \varphi}{1 - \psi}} \right) + \frac{N}{\lambda L + \rho N} \frac{\lambda}{1 + \frac{1 + \varphi}{1 - \psi}}
\]

\[
\leftrightarrow X_t = e^{\left( \frac{\lambda L}{N} + \rho \right) t} \left( X_0 - \frac{\lambda}{\lambda L + \rho N} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] \right) + \frac{\lambda}{\lambda L + \rho N} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right]
\]

Accordingly, one gets:

\[
L_t^Y = \frac{\lambda L + \rho N}{e^{\left( \frac{\lambda L}{N} + \rho \right) t} \left( \frac{\lambda L + \rho N}{L_0} - \lambda \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] \right) + \lambda \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right]}
\]

Using the transversality condition in the program of the representative household, we can show that \( L_t^Y \) immediately jumps to its steady-state level: \( L_t^Y = (\lambda L + \rho N) / \lambda \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] \). The transversality condition is only satisfied when \( L_t^Y = L_0^Y = \frac{\lambda L + \rho N}{\lambda (1 + \varphi) \alpha + 1 - \psi} \), \( \forall t \). Thus, one has \( g_{L_t^Y} = 0 \). Therefore, substituting into the system (30), one gets Lemma 1 below, in which we provide the complete characterization of the decentralized equilibrium. Note that, since \( \bar{g}_t \) is independent of time, \( \mathbb{E} [\bar{g}]_t = \mathbb{E} [\bar{g}], \forall t \).

**Lemma 1:** Assume \( L_t = L \) (i.e. \( n = 0 \)), \( N_t = N \), and \( \bar{g}_t = \bar{g}, \forall t \). One has \( \Omega_t = \Omega, \forall t \).

The repartition of labor at equilibrium is:

\[
L_t^Y = L_t^Y = \frac{(1 - \psi)(\lambda L + \rho N)}{\lambda [1 - \psi + (1 + \varphi) \alpha]}, \text{ and } l_{\omega t} = l = \frac{L (1 + \varphi) \alpha - (1 - \psi) \rho N}{N (1 + \varphi) \alpha + 1 - \psi}, \forall \omega \in \Omega
\]

The quantity of each intermediate good is:

\[
x_{\omega t} = x = \left( \frac{\alpha^2}{1 - \psi} \right)^{\frac{1}{(1 + \varphi) \alpha - (1 - \psi) \rho N}} L_t^Y, \forall \omega \in \Omega
\]
Given a vector of public policies \((\psi, \varphi)\), the equilibrium growth rates are:

\[
g_{t} = g_{c_{t}} = g_{x_{t}} = g(\psi, \varphi) = \frac{\lambda \sigma L}{N} \left( \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{\alpha N}{\alpha + 1 - \psi}}{(1 + \varphi)\alpha + 1 - \psi} \right) \left( p_{0} + E[\theta] \right)
\]

The prices are:

\[
r = g(\psi, \varphi) + \rho, \quad w_{t} = (1 - \alpha) \left( \frac{\alpha^{2}}{1 - \psi} \right)^{\frac{\omega}{\lambda}} K_{t}, \quad \text{and} \quad q_{\omega t} = q_{t} = \frac{K_{t}}{\alpha N}, \quad \forall \omega \in \Omega
\]

where \(E[\theta]_t = E[\theta] = p_{0} \Omega + p_{w} \bar{\Omega}, \forall t\).

### 6.2 Population Growth

In this subsection, we consider constant population growth \((g_{L_{t}} = n > 0)\). In order to allow for population growth, we introduce the commonly shared assumption of proportionality between the number of sectors and the population level \((N_{t} = \gamma L_{t})\). Under those assumptions, (33), becomes:

\[
g_{L_{t}Y} = \frac{\lambda}{\gamma L_{t}} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] L_{t}^{Y} = n - \rho - \frac{\lambda}{\gamma}
\]

(35)

As previously, in order to solve (35), let \(X_{t} = 1/L_{t}^{Y}\) (which implies \(g_{X_{t}} = -g_{L_{t}Y}\)). This gives the following first-order linear differential equation in \(X_{t}\):

\[
-g_{X_{t}} = \frac{\lambda}{\gamma L_{t}} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] 1 \frac{X_{t}}{L_{t}} = n - \rho - \frac{\lambda}{\gamma} \Leftrightarrow X_{t} = \left( \frac{\lambda}{\gamma} + \rho - n \right) X_{t} = -\frac{\lambda}{\gamma} \left[ 1 + \frac{1 + \varphi}{1 - \psi} \right] e^{-\alpha t}
\]

Its solution is:

\[
X_{t} = e^{(\frac{\lambda}{\gamma} + \rho - n) t} \left[ X_{0} = -\frac{\lambda}{\gamma + \rho} \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) + \frac{\lambda}{\gamma + \rho} \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) e^{-\alpha t} \right]
\]

\[
\Leftrightarrow X_{t} = e^{(\frac{\lambda}{\gamma} + \rho - n) t} \left[ X_{0} = \frac{\lambda}{\gamma + \rho} \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) + \frac{\lambda}{\gamma + \rho} \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) e^{-\alpha t} \right]
\]

Accordingly, one gets:

\[
L_{t}^{Y} = \frac{\lambda + \gamma \rho}{1 + \frac{\lambda + \gamma \rho}{\frac{\lambda}{\lambda + \gamma \rho}} - \lambda \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) + \lambda \left( 1 + \frac{1 + \varphi}{1 - \psi} \right) e^{-\alpha t}}
\]

Using the transversality condition in the program of the representative household, one obtains:

\[
L_{t}^{Y} = \frac{1 + \frac{\psi \alpha}{\alpha + 1 - \psi}}{1 + \frac{\psi \alpha}{\alpha + 1 - \psi}} L_{t} = \frac{(1 - \psi) \left( 1 + \frac{\psi \alpha}{\alpha + 1 - \psi} \right)}{(1 + \varphi)\alpha + 1 - \psi} L_{t} \quad \text{and} \quad g_{L_{t}Y} = n
\]

(36)

Plugging (36) in the system (30) gives the characterization of the decentralized equilibrium, given in Lemma 2 below. Note that, since the maximum diffusion of knowledge, \(\bar{\theta}_{t}\), is increasing in \(N_{t}\), then \(E[\theta]_t = E[\theta]_t, \forall t\).

**Lemma 2**: \(g_{L_{t}} = n > 0, N_{t} = \gamma L_{t}, \quad \text{and} \quad \bar{\theta}_{t} = \bar{\theta}(N_{t}), \forall t\).

The repartition of labor at equilibrium is:

\[
L_{t}^{Y} = \frac{(1 - \psi) \left( 1 + \frac{\psi \alpha}{\alpha + 1 - \psi} \right)}{(1 + \varphi)\alpha + 1 - \psi} L_{t} \quad \text{, and} \quad l_{\omega t} = l = \frac{(1 + \varphi)\alpha - (1 - \psi)\frac{\alpha N}{\alpha + 1 - \psi}}{(1 + \varphi)\alpha + 1 - \psi} \gamma, \forall \omega \in \Omega_{t}
\]

The quantity of each intermediate good is:

\[
x_{\omega t} = x_{t} = \left( \frac{\alpha^{2}}{1 - \psi} \right)^{\frac{\omega}{\lambda}} L_{t}^{Y}, \forall \omega \in \Omega_{t}
\]

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Given a vector of public policies \((\psi, \varphi)\), the equilibrium growth rates are:

\[
g_{\chi t} = g_{\nu t} = \lambda \sigma (p_0 + \mathbb{E} [\theta_t] t), \forall \omega \in \Omega_t
\]

\[
g_{\psi_t} = g_{\gamma_t} - n = g_{\chi_t} + n = g_t (\psi, \varphi) = \frac{\lambda \sigma}{\gamma} \left( \frac{(1 + \varphi) \alpha - (1 - \psi) \sigma}{\alpha + 1 - \psi} \right) (p_0 + \mathbb{E} [\theta_t]) + n \quad (37)
\]

The prices are:

\[
r_t = g_t (\psi, \varphi) + \rho, \quad w_t = (1 - \alpha) \left( \frac{\alpha^2}{1 - \psi} \right) \frac{K_t}{\alpha \gamma L_t}, \text{ and } q_{\omega t} = q_t = \frac{K_t}{\alpha \gamma L_t}, \forall \omega \in \Omega_t
\]

where \(\mathbb{E} [\theta_t] = p_n \theta + p_W \theta (\gamma L_t)\).

References


